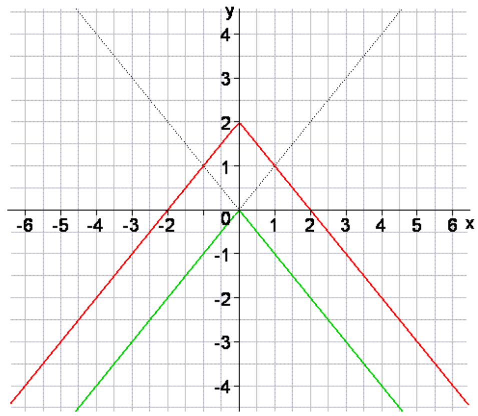
Combined transformations of graphs

Example 1: Sketch the graph of $y = -|x| + 2$.

Solution:

Reflecting the graph of $y = |x|$ in the x axis gives the graph of $y = -|x|$ (**green** graph).

Shifting the graph of $y = -|x|$ upwards through 2 units gives the graph of $y = -|x| + 2$ (**red** graph).



Example 2: Sketch the graph of $y = -x^2 - 4$.

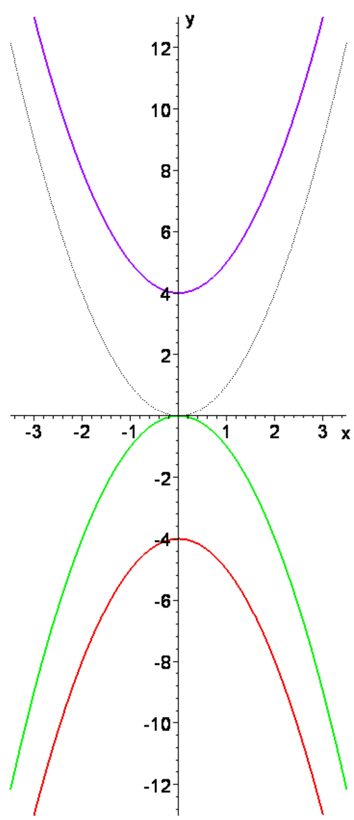
Solution:

Reflecting the graph of $y = x^2$ (dotted graph) in the x axis gives the graph of $y = -x^2$ (**green** graph).

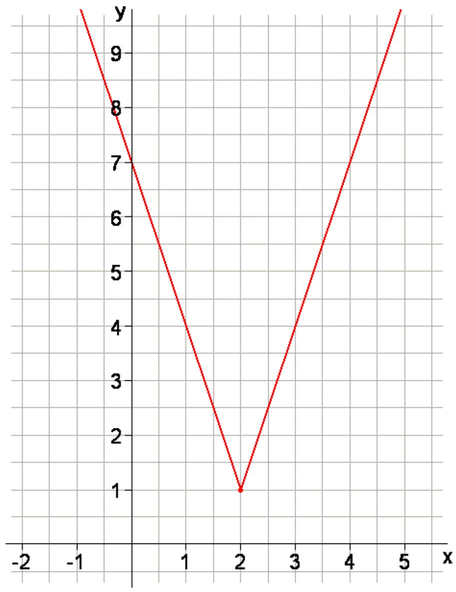
Shifting the graph of $y = -x^2$ downwards through 4 units gives the graph of $y = -x^2 - 4$ (**red** graph).

Alternatively, the graph of $y = x^2$ can be shifted upwards to give the graph of $y = x^2 + 4$ (**purple** graph) and then this graph can be reflected in the x axis

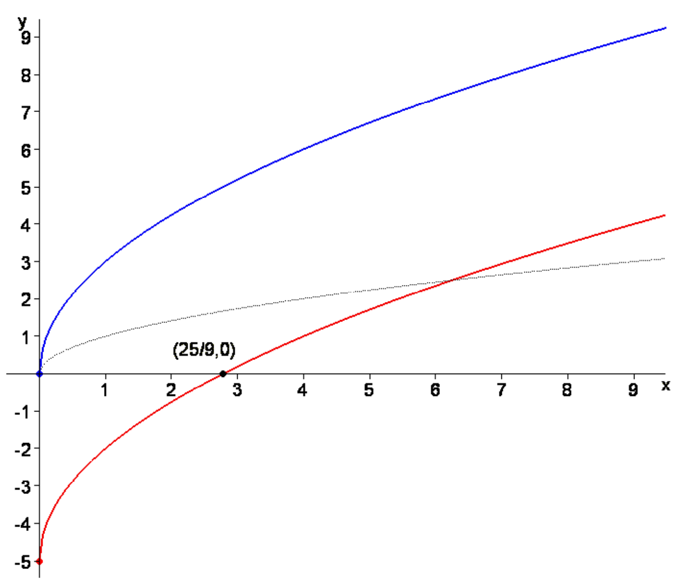
to give the graph of $y = -(x^2 + 4)$, which is equivalent to $y = -x^2 - 4$.



Example 3: Sketch the graph of $y = 3|x - 2| + 1$.
Solution:
The graph of $y = 3|x|$ can be obtained by stretching the graph of $y = |x|$ with a factor of 3 parallel to the y axis.
The graph of $y = 3|x - 2| + 1$ can be obtained by shifting the graph of $y = 3|x|$ horizontally through 2 units to the right and one unit vertically upwards.



Example 4: Sketch the graph of $y = 3\sqrt{x} - 5$.
Solution:
The graph of $y = 3\sqrt{x}$ (**blue** graph) can be obtained by scaling the y coordinates on the graph of $y = \sqrt{x}$ (dotted graph) with the factor 3 so that the graph stretches parallel to the y axis.
Then the graph of (**red** graph) can be obtained by shifting the graph of $y = 3\sqrt{x}$ through 5 units downwards.
The ***x* intercept** occurs where $3\sqrt{x} - 5 = 0$, that is, where $3\sqrt{x} = 5$, which is equivalent to $\sqrt{x} = \frac{5}{3}$, giving $x = \frac{25}{9} = 2\frac{7}{9} \simeq 2.7778$.



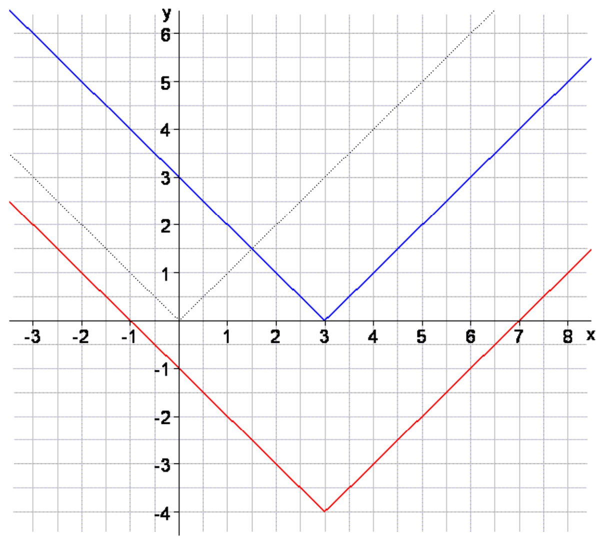
Example 5: Sketch the graph of $y = |x-3| - 4$.

Solution:

The graph of $y = |x-3|$ (**blue** graph) can be obtained by shifting the graph of $y = |x|$ (dotted graph) to the right 3 units.

Then the graph of $y = |x-3| - 4$ (**red**, graph) can be obtained by shifting the graph of $y = |x-3|$ downwards a distance of 4 units.

The ***x* intercepts** occurs where $|x-3| - 4 = 0$, that is, where $|x-3| = 4$, which is equivalent to $x - 3 = \pm 4$, giving $x = 3 + 4$, that is, $x = -1$ or $x = 7$. The ***y* intercept** is given by $y = |0-3| - 4 = 3 - 4 = -1$.



Example 6: Sketch the graph of $y = \dfrac{1}{x+3} + 2$.

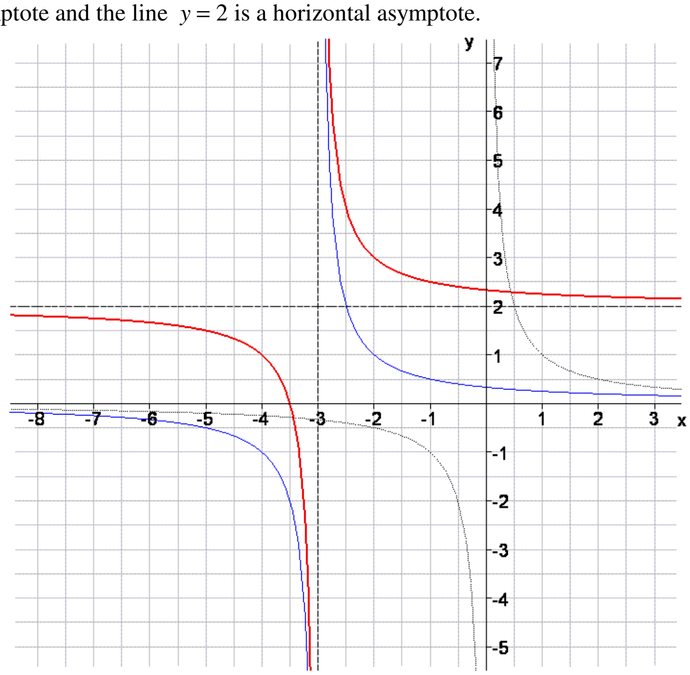
Solution:

The graph of $y = \dfrac{1}{x+3}$ (**blue** graph) can be obtained by shifting the graph of $y = \dfrac{1}{x}$ (dotted graph) through 3 units to the left.

Then the graph of $y = \dfrac{1}{x+3} + 2$ (**red** graph) can be obtained by shifting the graph of $y = \dfrac{1}{x+3}$ through 2 units upwards.

The ***x* intercept** occurs where $\dfrac{1}{x+3} + 2 = 0$, that is, where $\dfrac{1}{x+3} = -2$, which is equivalent to $x + 3 = -\dfrac{1}{2}$, giving $x = -\dfrac{7}{2} = -3.5$.

The line $x = -3$ is a vertical asymptote and the line $y = 2$ is a horizontal asymptote.



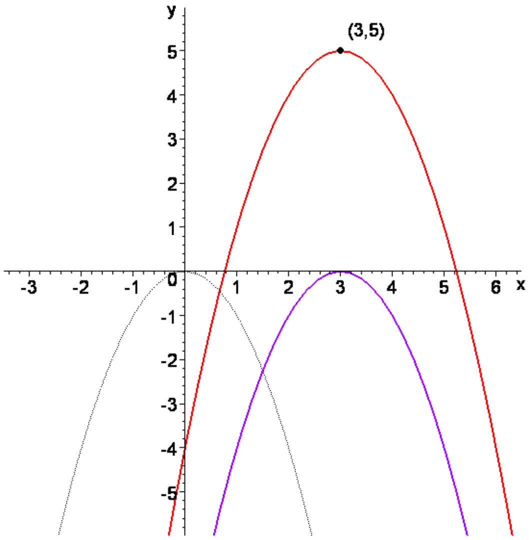
Example 7: Sketch the graph of $y = -(x-3)^2 + 5$.

Solution:

The graph of $y = -x^2$ (dotted graph) can be obtained by reflecting the graph of $y = x^2$ in the x axis.

The graph of $y = -(x-3)^2$ (**purple** graph) can be obtained by shifting the graph of $y = -x^2$ through 3 units to the right.

Then the graph of $y = -(x-3)^2 + 5$ (**red** graph) can be obtained by shifting the graph of $y = -(x-3)^2$ through 5 units upwards.



The ***x* intercepts** occur where $-(x-3)^2 + 5 = 0$, that is, where $-(x-3)^2 = -5$, which is equivalent to $(x-3)^2 = 5$, giving $x - 3 = \pm\sqrt{5}$, that is, $x = 3 \pm \sqrt{5}$. Decimal approximations for these x values are $x = 0.76393$ and $x = 5.23607$.

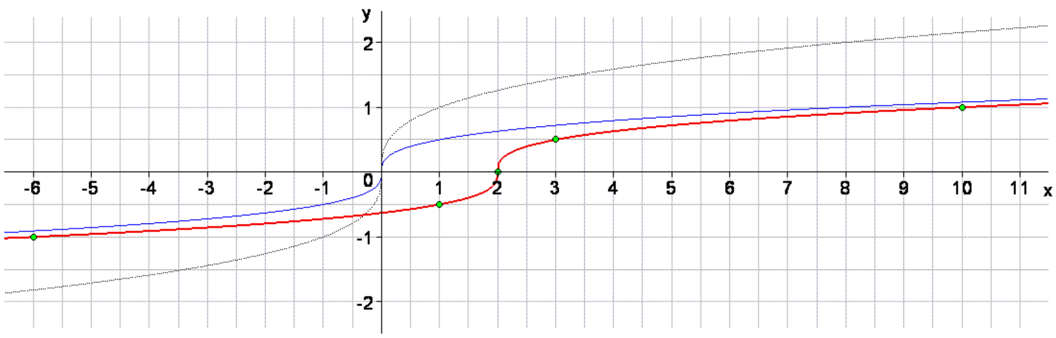
Example 8: Sketch the graph of $y = \frac{1}{2}\sqrt[3]{x-2}$.

Solution:

The graph of $y = \frac{1}{2}\sqrt[3]{x}$ (**blue** graph) can be obtained by scaling the y coordinates on the graph of $y = \sqrt[3]{x}$ with the factor $\frac{1}{2}$ so that the graph shrinks parallel to the y axis.

The graph of $y = \frac{1}{2}\sqrt[3]{x-2}$ (**red** graph) can be obtained by shifting the graph of $y = \frac{1}{2}\sqrt[3]{x}$ through a distance of 2 units horizontally to the right.

Note that the points $(-6, -1)$, $\left(1, -\frac{1}{2}\right)$, $(2, 0)$, $\left(3, \frac{1}{2}\right)$, $(10, 1)$ are on the graph of $y = \frac{1}{2}\sqrt[3]{x-2}$.



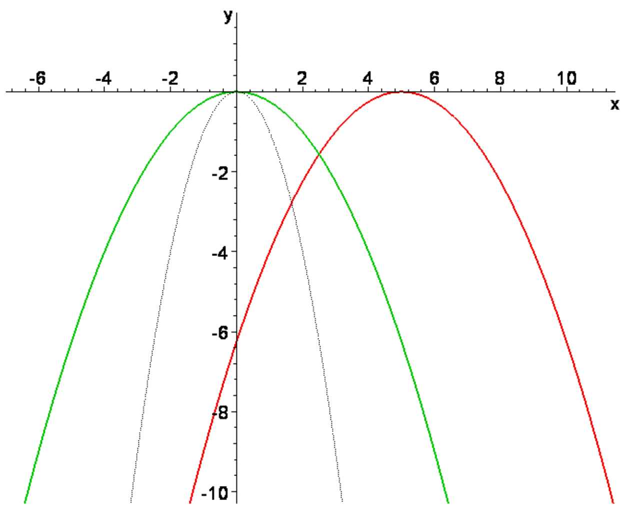
Example 9: Sketch the graph of $y = -\frac{1}{4}(x-5)^2$.

Solution:

The graph of $y = -x^2$ (dotted graph) can be obtained by reflecting the graph of $y = x^2$ in the x axis.

The graph of $y = -\frac{1}{4}x^2$ (**green** graph) can be obtained by scaling the y coordinates of points on the graph $y = -x^2$ with the factor $\frac{1}{4}$ which causes a shrinking parallel to the y axis.

Then the graph of $y = -\frac{1}{4}(x-5)^2$ (**red** graph) can be obtained by shifting the graph of $y = -\frac{1}{4}x^2$ through 5 units to the right.



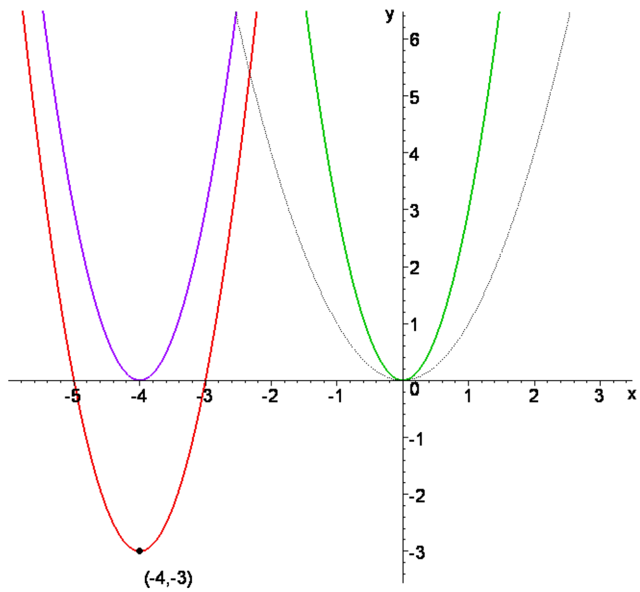
Example 10: Sketch the graph of $y = 3(x+4)^2 - 3$.

Solution:

The graph of $y = 3x^2$ (**green** graph) can be obtained by scaling the y coordinates on the graph of $y = x^2$ (dotted graph) with the factor 3 so that the graph stretches parallel to the y axis.

The graph of $y = 3(x+4)^2$ (**purple** graph) can be obtained by shifting the graph of $y = 3x^2$ through 4 units to the left.

Then the graph of $y = 3(x+4)^2 - 3$ (**red** graph) can be obtained by shifting the graph of $y = 3(x+4)^2$ through 3 units downwards.



The ***x* intercepts** occur where $3(x+4)^2 - 3 = 0$, that is, where $3(x+4)^2 = 3$, which is equivalent to $(x+4)^2 = 1$, giving $x + 4 = \pm 1$, that is, $x = -4 \pm 1$ so that $x = -5$ or $x = -3$.

Alternatively, $y = 3\,(x+4)^2 - 3$ is equivalent to $y = 3\,(x^2 + 8\,x + 15)$, that is, $y = 3\,(x+3)\,(x+5)$, so that we again see that the x intercepts occur where $x = -3$ or $x = -5$.

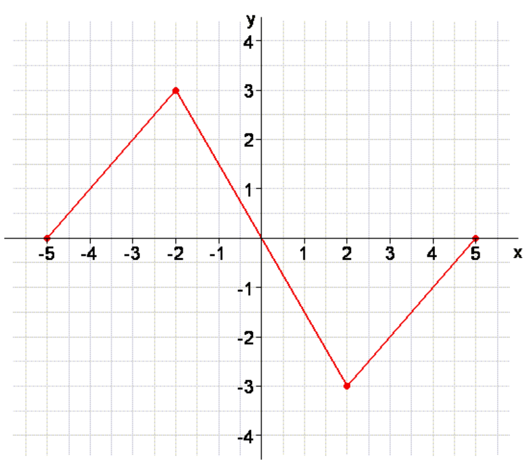
Example 11:
Suppose that the function f is defined by the piecewise formula:

$$f(x) = \begin{cases} x+5 & -5 \le x \textbf{ and } x < -2 \\ -\dfrac{3\,x}{2} & -2 \le x \textbf{ and } x < 2 \\ x-5 & 2 \le x \textbf{ and } x \le 5 \end{cases}.$$

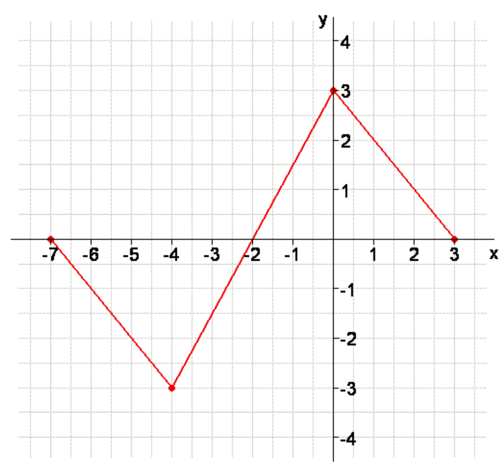
The domain of f is the closed interval $[-5, 5] = \{\, x \mid -5 \le x \le 5 \,\}$. The graph of f consists of three line segments. The first line segment joins the two points $(-5, 0)$ and $(-2, 3)$. This line has slope 1, and the point-slope equation of a line can be used to obtain its equation $y = x + 5$. Referring to the three part "recipe" for the function f, we use the formula $f(x) = x + 5$ when the input number x satisfies $-5 \le x < -2$. The second line segment joins the two points $(-2, 3)$ and $(2, -3)$. This line has slope $-\frac{3}{2}$. Since it passes through the origin, it has the equation $y = -\frac{3\,x}{2}$. f is evaluated using the formula $f(x) = -\frac{3\,x}{2}$ when the input number x satisfies $-2 \le x < 2$. The third line segment joins the two points $(2, -3)$ and $(5, 0)$. This line has slope 1, and it has the equation $y = x - 5$. f is evaluated using the formula $f(x) = x - 5$ when the input number x satisfies $2 \le x \le 5$.

We consider some transformations of the graph of f.

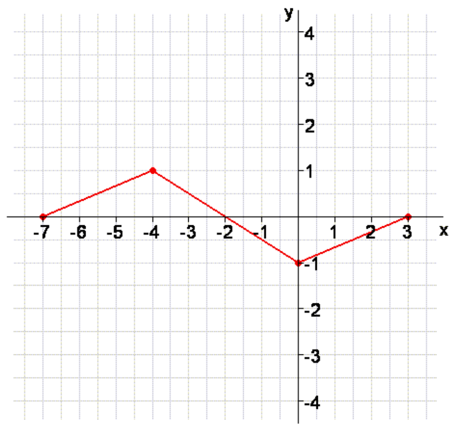
(a) $y = f(x)$
The domain of f is the closed interval $[-5, 5]$.
The range of f is the closed interval $[-3, 3]$.



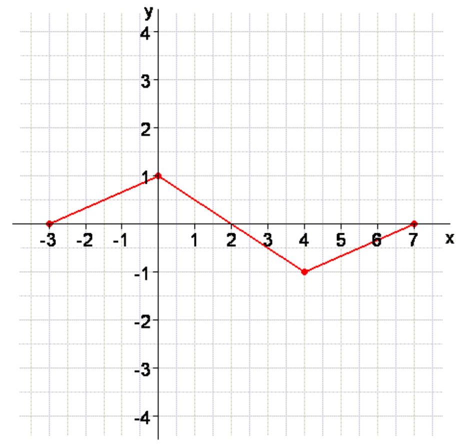
(b) $y = -f(x+2)$ First reflect the graph of of $y = f(x)$ in the x axis to give the graph of $y = -f(x)$, and then shift the graph of $y = -f(x)$ through 2 units to the left.



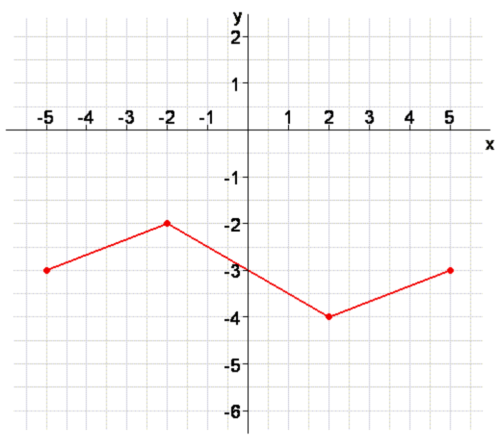
(c) $y = \frac{1}{3}\mathrm{f}(x+2)$ First scale the y coordinates on the graph of $y = \mathrm{f}(x)$ with the factor $\frac{1}{3}$ to give the graph of $y = \frac{1}{3}\mathrm{f}(x)$. Then shift this graph 2 units to the left.



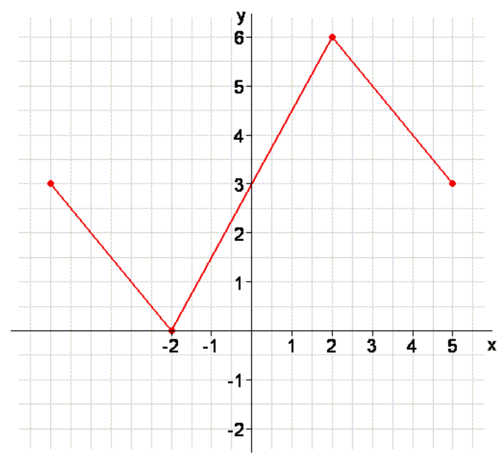
(d) $y = \frac{1}{3}\mathrm{f}(x-2)$ First scale the y coordinates on the graph of $y = \mathrm{f}(x)$ with the factor $\frac{1}{3}$ to give the graph of $y = \frac{1}{3}\mathrm{f}(x)$. Then shift this graph 2 units to the right.



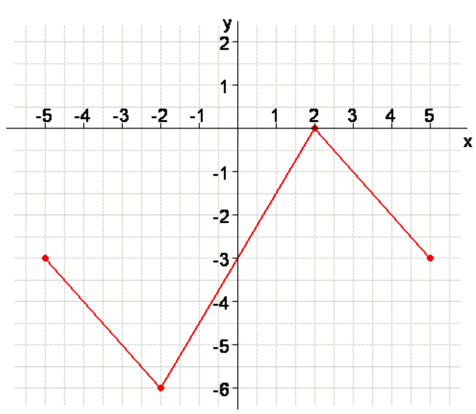
(e) $y = \frac{1}{3}\mathrm{f}(x) - 3$ First scale the y coordinates on the graph of $y = \mathrm{f}(x)$ with the factor $\frac{1}{3}$ to give the graph of $y = \frac{1}{3}\mathrm{f}(x)$. Then shift this graph 3 units down.



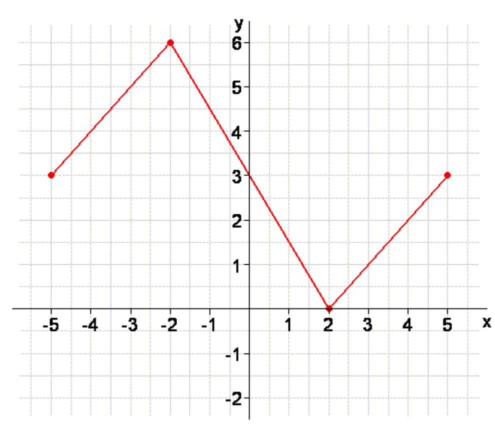
(f) $y = 3 - \mathrm{f}(x)$ First reflect the graph of of $y = \mathrm{f}(x)$ in the x axis to give the graph of $y = -\mathrm{f}(x)$. Then shift this graph 3 units up.



(g) $y = -\mathrm{f}(x) - 3$ First reflect the graph of of $y = \mathrm{f}(x)$ in the x axis. to obtain the graph of $y = -\mathrm{f}(x)$. Then shift this graph 3 units down.



(h) $y = \mathrm{f}(x) + 3$ Shift the graph of $y = \mathrm{f}(x)$ up 3 units.



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