

Second modified Prince-Dormand 13 stage combined order 7 and 8 Runge-Kutta scheme

See: High order embedded Runge-Kutta formulae, by P.J.Prince and J.R.Dormand,
Journal of Computational and Applied Mathematics, vol. 7, 1981, pages 67-75.

The nodes of the scheme are:

$$c_2 = \frac{2}{37}, c_3 = \frac{3}{37}, c_4 = \frac{9}{74}, c_5 = \frac{10384}{32623}, c_6 = \frac{22}{57}, c_7 = \frac{17}{112}, c_8 = \frac{19}{40},$$

$$c_9 = \frac{61483739}{100509189}, c_{10} = \frac{7}{11}, c_{11} = \frac{20159781079039}{21368476098509}, c_{12} = 1, c_{13} = 1.$$

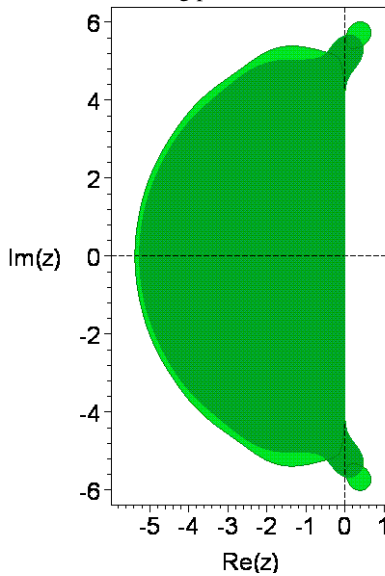
The principal error norm, that is, the 2-norm of the principal error terms is: $0.3941906126 \times 10^{(-5)}$.

The principal error norm of the order 7 embedded scheme is: $0.2512530052 \times 10^{(-4)}$.

The maximum magnitude of the linking coefficients is: 29.52262793.

The 2-norm of the linking coefficients is: 66.92537545.

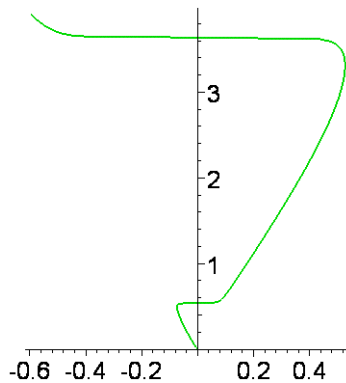
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 7 scheme appears in the darker shade.

The real stability intervals of the order 8 and 7 schemes are respectively $[-5.3777, 0]$ and $[-5.2578, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 8 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0.54274, 3.6376]$.

The coefficients in exact form are:

$c[2]=2/37,$
 $c[3]=3/37,$
 $c[4]=9/74,$
 $c[5]=10384/32623,$
 $c[6]=22/57,$
 $c[7]=17/112,$
 $c[8]=19/40,$
 $c[9]=61483739/100509189,$
 $c[10]=7/11,$
 $c[11]=20159781079039/21368476098509,$
 $c[12]=1,$
 $c[13]=1,$

$a[2,1]=2/37,$
 $a[3,1]=3/148,$
 $a[3,2]=9/148,$
 $a[4,1]=9/296,$
 $a[4,2]=0,$
 $a[4,3]=27/296,$
 $a[5,1]=1032390253115408/2812268013257727,$
 $a[5,2]=0,$
 $a[5,3]=-1308615948903296/937422671085909,$
 $a[5,4]=3788611045136896/2812268013257727,$
 $a[6,1]=4298723/124215768,$
 $a[6,2]=0,$
 $a[6,3]=0,$
 $a[6,4]=23731538336/124955009721,$
 $a[6,5]=55680556943/344905056072,$
 $a[7,1]=90425298380807/1941123694067712,$
 $a[7,2]=0,$
 $a[7,3]=0,$
 $a[7,4]=25858527697465277/224920823352164352,$
 $a[7,5]=-595015753921602442147/30824507001892830707712,$
 $a[7,6]=97514071661565/10223952112123904,$
 $a[8,1]=1608869893899153919357/62236848365206732800000,$
 $a[8,2]=0,$
 $a[8,3]=0,$
 $a[8,4]=52678163089953697185872159/265127722985892204000000000,$
 $a[8,5]=285744657987385238945380579837531/15158846468195614318853743411200000,$
 $a[8,6]=544687714617764129141319/2685782515876528640000000,$
 $a[8,7]=1578508482500088829214/54799140079494662109375,$
 $a[9,1]=998050699134281628870485668225039363078948627499982497/$
 $6193392961440825600382932686879139702424710849635698752,$
 $a[9,2]=0,$
 $a[9,3]=0,$
 $a[9,4]=-599451302040377804971492862879105869169196412602253649662/$
 $377703917334068424782470958844106772301281577518068638555,$
 $a[9,5]=-2815304417147833377529801173408466288979733164164893720470465821573/$
 $9448014668588380348762058120759656197134069734250628007541547801408,$
 $a[9,6]=-22372993307951847237583963822907850158523250144080904195/$
 $708909053211167058884601773446205680869979184308936408536,$
 $a[9,7]=77194923661114564990585902181681764267712245416241821495296/$
 $39152322781854614312506900607172775104474347794870492167465,$
 $a[9,8]=94414159384377879318340120427960985682000000/$
 $2386908673491277455905732718219769731737707011,$
 $a[10,1]=597587482413943255535090188496032537/2964717879563775095222516229596756544,$
 $a[10,2]=0,$

a[10,3]=0,
a[10,4]=-44894996211087045339352063094242/21622556805520708813249929943125,
a[10,5]=-70076783287799259184357131251119606363/206047243427505741120229833730802411712,
a[10,6]=-85786222354090999211933321628438441819/699854765273355256170212576244090272600,
a[10,7]=10908491490023847533473579773213358193635328/
4384009607065697673834338223011342339550625,
a[10,8]=89017854166145111783600000/1898015849746765308930700033,
a[10,9]=89352258620016864960365147286713638899678/
5405913631343191666415520877824548338886725,
a[11,1]=-2781115548912343773686813766668397639439673923539738021697178151007817179028548414569202949099974348752
0642829/146586727113873483028782220247975977708785595875113048252689062385172439741955499177697523231395080176
4947136,
a[11,2]=0,
a[11,3]=0,
a[11,4]=3970970096430749612788560891550894694302663035064894997505170565907901275521052676154726001886/
190451517277042351017053782209827410215805653533821363846152971021313522894764655071406902709,
a[11,5]=159248156206891430439240536810408886603840026407422125434835204000268580560466270014643221410536086223/

103117299549078740457342521235744762641331885553776811752069282933881429111884398382423084088129136192,
a[11,6]=-8494573315387392573892835461696880549487446399932265972922100257681483119286691666140631675014756983109
7019044725/2173095655673730050823709683164100895338312655329899212843413096874027322157199037933647542400261695
0016487280232,
a[11,7]=-7886992205784235730375610307527047506219530531504441917776160392971965280178124303127453101741823956875
864522084352/38150078929364065869237270477497802831576429062090471229069733738603160646702017701260553162052753
1035441092859929,
a[11,8]=42526709290208001807494134027837199956774332564690288734465198162972295260154057872445486593030000000/
36830623247590340440751560815598174054956779617183425373673177521908132768343752842009098664945230313,
a[11,9]=-3479635555790509817263364141028923429413587809327621220923513687451549555126933239343046819195145858716
23492585165645603868650/117863340757650693823952358683060994847437120623801793391257547953153747971372054476851
48055700978247977636054092933370023783,
a[11,10]=597967068482056180767410899396858498575970033471027842146097304084016645806427297996031822513843750/
25993275195371242675909719769637973022956182893962312066592166669729105381475345463689013271204287,
a[12,1]=3727579954156789107615026852154018033431275934320429/
2854681407100755462167282589266984620084206159685152,
a[12,2]=0,
a[12,3]=0,
a[12,4]=-222121238603351164796766875378/14843085607452922275955153125,
a[12,5]=-5555567717646938440152435775076367989/4455480097834156530857929450717595040,
a[12,6]=-6385463646780704607087670717351573833078187144085019125181/
2565659267050930919320554725395568173316158180082618696200,
a[12,7]=29915447363698934731566882948265241630108614546219155408896/
1878708164133600200047014792984372744381279855050506853125,
a[12,8]=3614752942069999191824588017865232696400000/
1287866764632264643869941874873543384632243,
a[12,9]=-2034310027481869012495617930148002640672401225198003396421389871591814305623/
474045009686163769342703668880746407928358172648236305879126065312630710500,
a[12,10]=10597978701544414628633886112952419/2759015469559174152715489456636680,
a[12,11]=4077869740819908844178631065125307798889700446270019961164777738495186924274824699667245379/
35597657593390285978504487947176829972707974938857070435889033997013272163491552758300973000,
a[13,1]=54303940804683937398259953457399150491899404875072536005810866816606375385360127/
23693749016118720454593631926555249286441054273822725843476854735324082832000000,
a[13,2]=0,
a[13,3]=0,
a[13,4]=-97289387109499681585649062900107345484277971048272715219/
3849907998193789784744147884340788179393113573876953125,

a[11,9]=-29.52262793013222162626896477481290109523663099084452326264295139159962536069058023591,
a[11,10]=23.00468348015410070042088692949932825311344784941323397561059337646334327926054557495,
a[12,1]=1.305777921446778403896271663337508206403268277289023481082147913305918351365795199218,
a[12,2]=0.,
a[12,3]=0.,
a[12,4]=-14.96462692984947573164526822671184861388233684517466010028722848927344076818348437227,
a[12,5]=-1.246906639836084789756171048485975644386355898232215986823801996881293157479297123521,
a[12,6]=-2.488819824512553531295558695698125218012490435084333661330017491182461420376031225035,
a[12,7]=15.92341372375681200938326901986161197627286984512505480228406256201816873170577881716,
a[12,8]=2.806775546461244224265428705353662189219833434186667860926360180007532724273016609142,
a[12,9]=-4.291385809184367026975457880306349741874487497451657457589433257034500132859828781169,
a[12,10]=3.841217571439613065731705901030298210569019367838756665648624453615951704417917158354,
a[12,11]=.1145544402780333763957805616192186356906797515033643960892861254241239671361337181191,
a[13,1]=2.291910020982381490703017103301227886278179102551383371171613218008503004963505053002,
a[13,2]=0.,
a[13,3]=0.,
a[13,4]=-25.27057455792285205193317799355495644053824620319623930267666459358343599006051660213,
a[13,5]=-1.955967142402501301180372708873731618188368235418975650791548633464915701319984435374,
a[13,6]=.4024922725872925758192848211971775448214124407788145603057256725188091394054243314282e-1,
a[13,7]=26.13964564835776332161101732733126554363249527309474510402631698255266494847319085481,
a[13,8]=-3.615882988674772025456841096076614417333106300447842776105663106236745966649548454913,
a[13,9]=10.62468485286290805383023361602767792173043833992561088690920121156565008074016430625,
a[13,10]=-7.408058681771399067672894813127643517007584439198615570119893687077337223075560002671,
a[13,11]=.1539936213097423225170900828530568869440512186120524815560660409837359329882068478882,
a[13,12]=0.,

b[1]=.4291605240700781507143988938317269408198766712424547188218590475521246390980140787487e-1,
b[2]=0.,
b[3]=0.,
b[4]=0.,
b[5]=0.,
b[6]=- .7026198155628613844968614533252819215242890395491316974533046047568480619736341277627e-1,
b[7]=.2466686024581284783947219193297469805861873178849917561643526305107121765406121812281,
b[8]=.7023383493764024474713026706791966521163207047117187375744715002839801895816721924616,
b[9]=-2.413675774116385360701995348772388011760036317326687593860260070705546422875255840156,
b[10]=2.338535890458838082517847117189940321564471343260348904617404450455664956551636060796,
b[11]=.1603650439732572363170523440785968361878842172676395066594873946367658023169575582554,
b[12]=- .2568861830009625606206824465557372806243860289673436132923113494611043598280601476842,
b[13]=.25,

b*[1]=.3353048714878325047977380580482489242328602573513746932923869425680813935840220112879e-1,
b*[2]=0.,
b*[3]=0.,
b*[4]=0.,
b*[5]=0.,
b*[6]=- .6862293894579303872129878918234572322731216694268535695619507721998529741474294627893,
b*[7]=.3024558012941764893878536534414853388814806347008515030045308135911685965351138092621,
b*[8]=1.995653799948371347719367254245457323036045674095521365914494065850625000996363241023,
b*[9]=-5.893884035440639219306759845663475218075942150070225999099628033171009807004796093815,
b*[10]=5.136428749097814435970891734803167319659365983075386838437048631173857561774497434270,
b*[11]=.7500755037238704592482425215496053931184846485314535493922956346136644545081183388315e-1,
b*[12]=.3704e-1,
b*[13]=0.