An efficient 13 stage combined order 7 and 8 Runge-Kutta scheme

This scheme could be regarded as another modification of Jim Verner's "Maple" scheme.

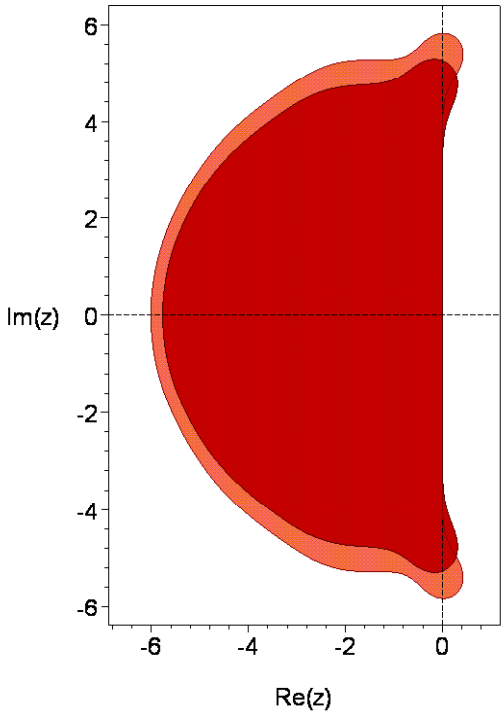
The nodes of the scheme are:

$$c_2=\frac{27}{400},\ c_3=\frac{469}{4510},\ c_4=\frac{1407}{9020},\ c_5=\frac{100}{257},\ c_6=\frac{7}{15},\ c_7=\frac{7}{45},\ c_8=\frac{58}{59},\ c_9=\frac{49117478}{55144743},\ c_{10}=\frac{13}{14},\ c_{11}=\frac{62}{65},\ c_{12}=1,\ c_{13}=1.$$

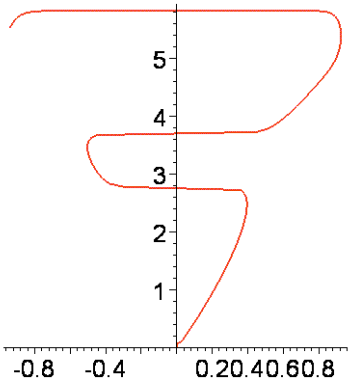
The principal error norm, that is, the 2-norm of the principal error terms is: $0.5733954042 \times 10^{(-6)}$.

The principal error norm of the order 7 embedded scheme is: $0.1003858680 \times 10^{(-4)}$.
The maximum magnitude of the linking coefficients is: 18.09864768.
The 2-norm of the linking coefficients is: 55.61025323.

The stability regions for the two schemes are shown in the following picture.



The stability region of the order 7 scheme appears in the darker shade.
The real stability intervals of the order 8 and 7 schemes are respectively $[-6.0124, 0]$ and $[-5.7679, 0]$.
The following picture shows the result of distorting the boundary curve of the stability region of the order 8 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the union of intervals: $[0, 2.7703] \cup [3.7022, 5.8244]$.

The coefficients in exact form are:
c[2]=27/400,
c[3]=469/4510,
c[4]=1407/9020,
c[5]=100/257,
c[6]=7/15,
c[7]=7/45,
c[8]=58/59,
c[9]=49117478/55144743,
c[10]=13/14,
c[11]=62/65,
c[12]=1,
c[13]=1,

a[2,1]=27/400,
a[3,1]=1311793/54918270,
a[3,2]=439922/5491827,
a[4,1]=1407/36080,
a[4,2]=0,
a[4,3]=4221/36080,
a[5,1]=12985296430100/33603736057857,
a[5,2]=0,
a[5,3]=-16218027650000/11201245352619,
a[5,4]=48744170200000/33603736057857,
a[6,1]=3817807/81405000,
a[6,2]=0,
a[6,3]=0,
a[6,4]=10274191312/43991343405,
a[6,5]=40755997793/218862405000,
a[7,1]=8883007/146529000,
a[7,2]=0,
a[7,3]=0,
a[7,4]=53436069013712/475502430864645,
a[7,5]=-4116355777093/117791746371000,
a[7,6]=7635614/436305285,
a[8,1]=-257501551162165951214471917239710702741532800 0,
a[8,2]=0,
a[8,3]=0,
a[8,4]=-743/55,
a[8,5]=-319343831168452865834105090 3/315992014923229256968928000,
a[8,6]=1833349835920600825632572293018694000478873228 8,
a[8,7]=1858648707423022382406206 1/10356928917172162097601856 0,
a[9,1]=-92212217189471655221780797327793420467234876635197955068071590 1/
157401190198671640845526912912949987653984663002982764253660800 0,
a[9,2]=0,
a[9,3]=0,
a[9,4]=-24939241129656091104224344877438120589127765764453317410687 63/
248576188041998999363340039748303974885433202092103843801085,
a[9,5]=-34741045425295020689360906080273644148743557644527553388495035454775694 9/
69747362766091883481592212536178438630509515521207481681627188467808000,
a[9,6]=53277337079395844786125848105303201123146656324055860158220994 59/
12498089552051473201987367363212255968282639800930482622603723 52,
a[9,7]=63805647563182428528344562389882620890969933364805025903916023/
52225368801254480919325372124301691131185132948965843160633 60,
a[9,8]=16756974250619222216361657668160522447776912/16497659305082807158860334847772546655402280 07,
a[10,1]=-37029278245341097823148049405328590743644 87/413125770345999630231669625105519038156800 0,
a[10,2]=0,
a[10,3]=0,

a[10,4]=-485424949192857745894089772363 97/3795156546073735065277428293920,
a[10,5]=-22919425542674277431989520078944496953957193/319462007754808947240877762440203865804 8000,
a[10,6]=2484540315797359406715529397919751486619672787/42164869038880297436703627063209609819 0348288,
a[10,7]=17054697614357642541421105633253210641175183343/10721807404878036207849378626628453013 29326080,
a[10,8]=24246496168069054546750207 5/123964886651855 71302473970944,
a[10,9]=-27672065893024425024193234139018163829008511 5/96637936155048616171937073723046688059964 12672,
a[11,1]=-88826526149223807070829373027939306268646309739 71/
80180429743401295397733653580724007953352654400 00,
a[11,2]=0,
a[11,3]=0,
a[11,4]=-913007740298173200319620395918956 93/66417762608855977419301709950835 25,
a[11,5]=-11220125505347440190681671636785097372524556302 9/
13045194205836605228092670613416996666049440000,
a[11,6]=18333480904050081466203638124283780761857633059295421 3/
26459846979744846747689472891117696020043251727064960,
a[11,7]=59085344439790277683397952622868924535489307749240825 27/
33780167695209416728062261777288976098597019505562 5600,
a[11,8]=66124495037696378406900160469675 66/22555686877706028458625838437394237 5,
a[11,9]=-10293784278236522497308869863513789970987031746931000834/
87878263633955728892467733863749576402441977455055861876 75,
a[11,10]=-42156296379564046245655 04/10812361259129223437650012 5,
a[12,1]=-53530873684297375488831160483367419094955927 7/
50766547052606143813695530390024885424662360 00,
a[12,2]=0,
a[12,3]=0,
a[12,4]=-65813855002186524202842983444063 12/84246402888837801802727580865321 5,
a[12,5]=-22981099062884908032932152837250718690686309 7/
29548105906232803976194189596626971177604000,
a[12,6]=33362174362860446465153593700072886543606354183 3/
54042431430277972080199874664118662613066091504,
a[12,7]=28143622302890122836076724075705459349698417419989 3/
24784655195828222920388514383970641394482546714960,
a[12,8]=-19581608724166102327634239 1/39989664801267468527787510 6,
a[12,9]=14464527008223808452459454468398839957564434825455126796 5/
31485646682871080763588948318135902300118760332187063997 3,
a[12,10]=-11336555587761851927258867 2/93091504282232088010468779,
a[12,11]=90084630318648875570943 75/66068623478006805843316 78,
a[13,1]=-35556891109301815175584262675897743594621/233372257669869909529298272528632296280 00,
a[13,2]=0,
a[13,3]=0,
a[13,4]=-15763198989262628277502495250456/12005621928416165478102975660 45,
a[13,5]=-47449922969007504396685334749052234618191 1/
42107837966579029834778387168434144652000,
a[13,6]=67954598770123713879003576483596707674131067 9/
77013733239195245746002612595738122195430352,
a[13,7]=13688171021909762189121898479960578456471695 77/
75630904954724828436404430778019911819343440,
a[13,8]=-32924968721657273226732 58/59253630028479896512047753,
a[13,9]=-26845223094905765297353027274273076760309620093 90/
34628336630650248957306474374479340566374128519867,
a[13,10]=64631913714031794733568/48642375362430299880264 9,
a[13,11]=0,
a[13,12]=0,

b[1]=1049982578583103/23627320335287760,
b[2]=0,
b[3]=0,
b[4]=0,

b[5]=0,
b[6]=768239620641637875/2161676880469378496,
b[7]=7884214609987698755625/31681975342943333980208,
b[8]=-19726902451409487557291/9259475236790114044800,
b[9]=22985448977257586298821077777442163519710451498500431391 61/
732352661995952143676542507482891823353467854709785979600,
b[10]=-2307497857833742448/278745570101106225,
b[11]=14896366464322825 1375/20207195606693950848,
b[12]=576091109999/2308683585600,
b[13]=0,

b*[1]=4850006989829/108881660531280,
b*[2]=0,
b*[3]=0,
b*[4]=0,
b*[5]=0,
b*[6]=5801298400186875/16253209627589312,
b*[7]=2408095926060530625/9691641279578872432,
b*[8]=5500622985780479 2033/15747406865289309600,
b*[9]=3634848925697241080953525086368997029484485574 63/
160778106318887934237641558033288694994268792800,
b*[10]=-14197733827415152/4039790871030525,
b*[11]=0,
b*[12]=0,
b*[13]=-9030611407/4779883200.

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