

**Enright-Verner 13 stage combined order 7 and 8 Runge-Kutta scheme**

See: The Relative Efficiency of Alternative Defect Control Schemes for High-Order Continuous Runge-Kutta Formulas  
 W. H. Enright SIAM Journal on Numerical Analysis, Vol. 30, No. 5. (Oct., 1993), pp. 1419-1445.

The nodes of the scheme are:

$$c_2 = \frac{139}{2500}, c_3 = \frac{473499}{4616000}, c_4 = \frac{1420497}{9232000}, c_5 = \frac{1923}{5000}, c_6 = \frac{923}{2000}, c_7 = \frac{769}{5000},$$

$$c_8 = \frac{8571}{10000}, c_9 = \frac{14280181565693441800}{15023510624554266963}, c_{10} = \frac{3611}{5000}, c_{11} = \frac{15}{16}, c_{12} = 1, c_{13} = 1.$$

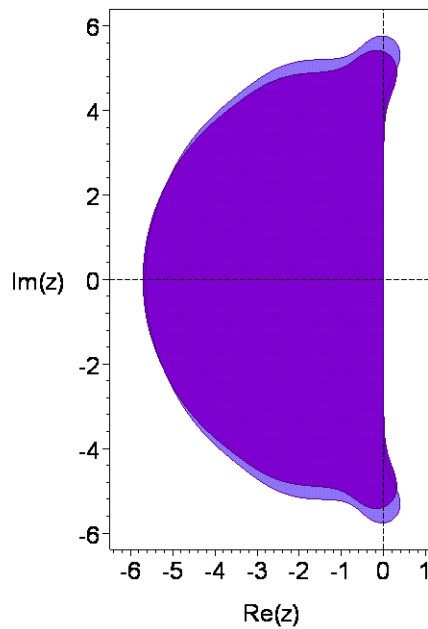
The principal error norm, that is, the 2-norm of the principal error terms is:  $0.1295525309 \times 10^{(-5)}$ .

The principal error norm of the order 7 embedded scheme is:  $0.2723687442 \times 10^{(-4)}$ .

The maximum magnitude of the linking coefficients is: 19.18139263.

The 2-norm of the linking coefficients is: 50.73279983.

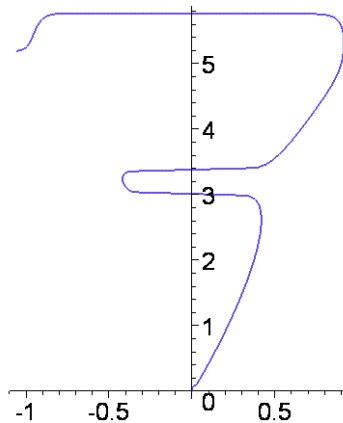
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 7 scheme appears in the darker shade.

The real stability intervals of the order 8 and 7 schemes are respectively  $[-5.6426, 0]$  and  $[-5.7009, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the order 8 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the union of intervals:  $[0, 3.0015] \cup [3.3817, 5.7604]$ .

The coefficients in exact form are:

$c[2]=139/2500,$   
 $c[3]=473499/4616000,$   
 $c[4]=1420497/9232000,$   
 $c[5]=1923/5000,$   
 $c[6]=923/2000,$   
 $c[7]=769/5000,$   
 $c[8]=8571/10000,$   
 $c[9]=14280181565693441800/15023510624554266963,$   
 $c[10]=3611/5000,$   
 $c[11]=15/16,$   
 $c[12]=1,$   
 $c[13]=1,$

$a[2,1]=139/2500,$   
 $a[3,1]=94226774499/11846945536000,$   
 $a[3,2]=224201303001/2369389107200,$   
 $a[4,1]=1420497/36928000,$   
 $a[4,2]=0,$   
 $a[4,3]=4261491/36928000,$   
 $a[5,1]=3590880889914277/9341720958375000,$   
 $a[5,2]=0,$   
 $a[5,3]=-1122077307242443/778476746531250,$   
 $a[5,4]=1683359084698258/1167715119796875,$   
 $a[6,1]=3279216863/71027928000,$   
 $a[6,2]=0,$   
 $a[6,3]=0,$   
 $a[6,4]=1418466643672/6146756958375,$   
 $a[6,5]=725783021041/3932390759616,$   
 $a[7,1]=22628776819090891/378192197956950000,$   
 $a[7,2]=0,$   
 $a[7,3]=0,$   
 $a[7,4]=116326878837670912/1047320101758291075,$   
 $a[7,5]=-21022506263989/614436056190000,$   
 $a[7,6]=182198401/10649112500,$   
 $a[8,1]=-150702609045465151/280142368857000000,$   
 $a[8,2]=0,$   
 $a[8,3]=0,$   
 $a[8,4]=-47048174572430533795112/6781573975400382234375,$   
 $a[8,5]=-29670244019740727/6363654238520000,$   
 $a[8,6]=503350581600824913/125990342179812500,$   
 $a[8,7]=9,$   
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 $33244146421522730587295739613,$   
 $a[9,2]=0,$   
 $a[9,3]=0,$   
 $a[9,4]=-23585391478818778730394458117511032898674036917053932712809944820432633202719845187666625892292454539196$   
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a[10,4]=35076261889213578261995286390053983221937920015616/  
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a[10,5]=29877053472248545227782869189767925950557009/  
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b[1]=55284231195707975647126708111723/1258814250475014554517603612114000,  
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