

**An 11 stage, order 7 Runge-Kutta scheme with an order 6 embedded scheme**

See: Explicit Runge-Kutta Pairs with One More Derivative Evaluation than the Minimum, by P.W.Sharp and E.Smart, Siam Journal of Scientific Computing, Vol. 14, No. 2, pages. 338-348, March 1993.

The nodes of the scheme are:

$$c_2 = \frac{1}{200}, c_3 = \frac{93}{479}, c_4 = \frac{110}{241}, c_5 = \frac{509}{845}, c_6 = \frac{244}{273}, c_7 = \frac{222785849}{1109145221}, c_8 = \frac{17}{20}, c_9 = \frac{171}{181}, c_{10} = 1, c_{11} = 1.$$

The principal error norm of the order 7 scheme, that is, the 2-norm of the principal error terms is:  $0.8291209743 \times 10^{(-5)}$ .

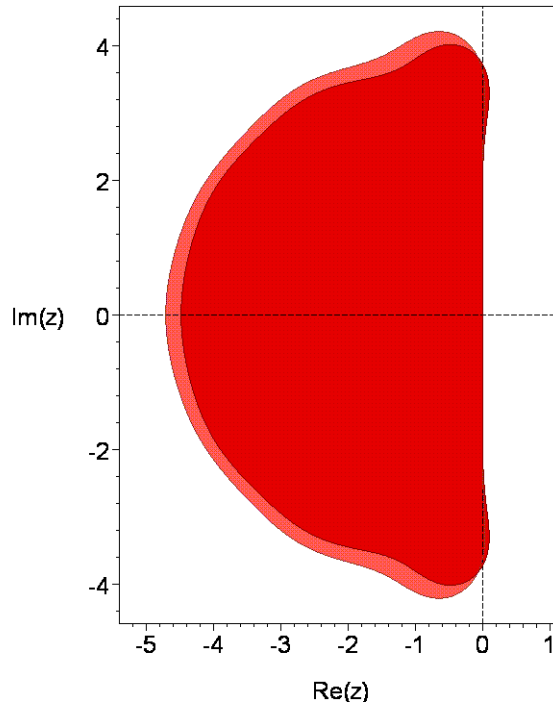
The 2-norm of the order 9 error terms is  $0.5673310328 \times 10^{(-4)}$ , which is approximately 6.843 times the principal error norm.

The principal error norm of the order 6 embedded scheme is:  $0.6140225088 \times 10^{(-4)}$ .

The maximum magnitude of the linking coefficients is: 9.486339238.

The 2-norm of the linking coefficients is: 19.31411741.

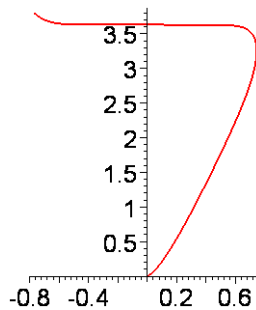
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 6 scheme appears in the darker shade.

The real stability intervals of the order 7 and 6 schemes are respectively  $[-4.70415, 0]$  and  $[-4.4759, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the order 7 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval:  $[0, 3.6334]$ .

The coefficients in exact form are:

$c[2]=1/200,$   
 $c[3]=93/479,$   
 $c[4]=110/241,$   
 $c[5]=509/845,$   
 $c[6]=244/273,$   
 $c[7]=222785849/1109145221,$   
 $c[8]=17/20,$   
 $c[9]=171/181,$   
 $c[10]=1,$   
 $c[11]=1,$

$a[2,1]=1/200,$   
 $a[3,1]=-820353/229441,$   
 $a[3,2]=864900/229441,$   
 $a[4,1]=-432520/5401533,$   
 $a[4,2]=0,$   
 $a[4,3]=2897950/5401533,$   
 $a[5,1]=311712996187/3366699277500,$   
 $a[5,2]=0,$   
 $a[5,3]=996040375149076/5096677701243375,$   
 $a[5,4]=344546810496217/1096059720697500,$   
 $a[6,1]=-228/1231,$   
 $a[6,2]=0,$   
 $a[6,3]=15089501252616323812/14457231345452578137,$   
 $a[6,4]=-6255616188978805360/5200809295819122531,$   
 $a[6,5]=103959712518200500/83969153915827131,$   
 $a[7,1]=20784021508544698439004341302945788046687497048823/$   
 $333809007739010923825478541340748044990946025336204,$   
 $a[7,2]=0,$   
 $a[7,3]=5908707677638153633449865101135634541623011524075277164/$   
 $31545554087781255456304827404537875875692253868011534003,$   
 $a[7,4]=-48597609826899455036511090495924959702383070843517685/$   
 $488091226723681375828622960423688756917868034622078292,$   
 $a[7,5]=40837242552907904940455804285850036846494300434360648500/$   
 $734372562662145990989587253914642887902719996408150162943,$   
 $a[7,6]=-2267236012274582026887011527795707923286/477373186945391376596029499088911158349111,$   
 $a[8,1]=-14850375572183825804482268314325533/131869870061161226458875010296840000,$   
 $a[8,2]=0,$   
 $a[8,3]=25892910773607576734414005208753/10529291320508273201747374752000,$   
 $a[8,4]=-613522852101531567576566644397/814576707522236069963552640000,$   
 $a[8,5]=8122715759499790031095209746537365648167969/8362965435593003722836700426272322940782592,$   
 $a[8,6]=-42937178562595251680361031143/2273295839722617724238904704000,$   
 $a[8,7]=-7424205995886059956357359508852528825590423/4378197291051651257486258563957705093888000,$   
 $a[9,1]=-153831689800158592252487409542055462465307763/517713869734060023599533102763919424014608068,$   
 $a[9,2]=0,$   
 $a[9,3]=-1334984216509015564184248020843701584/667067642557939663406130018977478253,$   
 $a[9,4]=-57633497085298645064371869898744017/30963779845911010423286154735455876,$   
 $a[9,5]=489177972756706421411944299196898256388191463578294580375/$   
 $319291201100986622665402847780351230046376607122311276442,$   
 $a[9,6]=-1185679300358377756233489606885369481439862/18438601757968638166315261704657797344684897,$   
 $a[9,7]=11944482045828658346298806908238121417524848379605193796762013/$   
 $3412376317311298392350722812851096867170775526794634954003706,$   
 $a[9,8]=103973108798709992837929472000/762442644104164301247937014633,$   
 $a[10,1]=-40734473647257020156088313054846628592379/112331307637222010640347265020564130793284,$   
 $a[10,2]=0,$   
 $a[10,3]=-1111042984674821389643230968704444252/147519964335786292056835317593798019,$   
 $a[10,4]=-5213063483277854003640774248205317/2282514999055829721733809658986516,$

a[10,5]=4344125602637174263526484423723747582563035478367000/  
281969717471900849409975568523101393352915992886809,  
a[10,6]=142812438169955950969368011788034519784/281984196771808918111529577339005012981,  
a[10,7]=35011829759852921774894466285700128707491070752921128030483969446296/  
3718044938427279904428695460050071040231420419464687750808112491477,  
a[10,8]=-26328105087975815319887872000/966150763091150023719018517747,  
a[10,9]=-1198575787406124991100974435169141280/4635648233480921649154730958571956313,  
a[11,1]=-4549513437278902549066809605929617243853/11637130531712243439602356681247846961408,  
a[11,2]=0,  
a[11,3]=-8070051946566656273895465364457/1073987714691843866578232224704,  
a[11,4]=-10059066143857143308915950403/4154341208363403956814118464,  
a[11,5]=5815991882243812040956722134361657936380202840699902125/  
3585951737451764935284196541594543916113759295056049408,  
a[11,6]=28274787710189875324117065188869812987/61207302919331402809078602199860439808,  
a[11,7]=5719170472909759253773842353503580127446876084974327902851679027589/  
602884877866360860548455552363583579732315378471940789964794317568,  
a[11,8]=1203167959926676490500/125196265101038882839767,  
a[11,9]=-4684487619570658562655565/18486842815356938821237824,  
a[11,10]=0,

b[1]=17499592155841793/301627899585331605,  
b[2]=0,  
b[3]=0,  
b[4]=0,  
b[5]=89929293580433773826711625625/175034206955063764470139052544,  
b[6]=43965679317071687650068867/11554641620922992752054730,  
b[7]=639711292154532559948032664549872463482448777204372657657897547/  
2011601425501588622427604443016911932371008211435797382991660160,  
b[8]=-43614217354789120000/21595629876813903267,  
b[9]=-210996982649291857501475159/99036657939412172256631200,  
b[10]=7077278529846113/15546034297382400,  
b[11]=0,

b\*[1]=94288552637982437/1608682131121768560,  
b\*[2]=0,  
b\*[3]=0,  
b\*[4]=0,  
b\*[5]=532400445239427668047625/994151030051934322008696,  
b\*[6]=15979368227021270883046053/3187487343702894552290960,  
b\*[7]=683482552868822360583731464535267340427333118635079/  
2165562941739167798045518397620231220712261142369860,  
b\*[8]=-132403130312946592000/50389803045899107623,  
b\*[9]=-169578451828457839968131453/57771383797990433816368200,  
b\*[10]=0,  
b\*[11]=16/25.