## A 7 stage, combined order 6 and 4 Runge-Kutta scheme

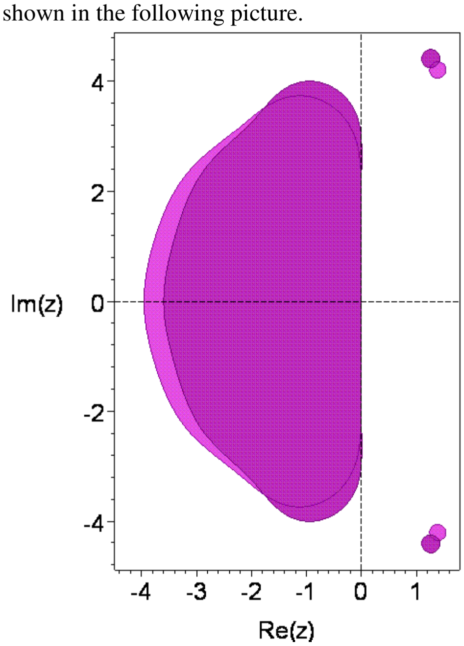
See: (1) A general four-parameter non-FSAL embedded Runge–Kutta algorithm of orders 6 and 4 in seven stages,
by M.E.A. El-Mikkawy and M.M.M. Eisa,
Applied Mathematics and Computation, Vol. 143, No. 2, (2003) pages 259 to 267.
(2) Cheap Error Estimation for Runge-Kutta methods, by Ch. Tsitouras and S.N. Papakostas,
Siam Journal on Scientific Computing, Vol. 20, Issue 6, Nov 1999.

The scheme considered here is a minor modification of a scheme described in the second paper.
The nodes of the scheme are:

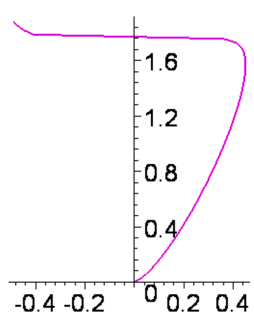
$$c_2 = \frac{4}{27},\ c_3 = \frac{2}{9},\ c_4 = \frac{3}{7},\ c_5 = \frac{11}{16},\ c_6 = \frac{10}{13},\ c_7 = 1.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.2117170563 \times 10^{(-3)}$.
**Note**: The order 6 scheme satisfies 2 of the 48 principal error conditions including the order 7 quadrature condition.
The principal error norm of the order 5 embedded scheme is: $0.8491158840 \times 10^{(-3)}$.
The maximum magnitude of the linking coefficients is: $\frac{6597591}{7972456} \simeq 0.8275481232$.
The 2-norm of the linking coefficients is: 1.962044023.

The stability regions for the two schemes are shown in the following picture.




The stability region of the order 5 scheme appears in the darker shade.
The real stability intervals of the order 6 and 5 schemes are respectively $[-3.9541, 0]$ and $[-3.5959, 0]$.
The following picture shows the result of distorting the boundary curve of the stability region of the order 6 scheme horizontally by taking the 11th root of the real part of points along the curve.




The stability region intersects the nonnegative imaginary axis in the interval: $[0, 1.7644]$.

The coefficients are:
c[2]=4/27,
c[3]=2/9,
c[4]=3/7,
c[5]=11/16,
c[6]=10/13,
c[7]=1,

a[2,1]=4/27,
a[3,1]=1/18,
a[3,2]=1/6,
a[4,1]=66/343,
a[4,2]=-729/1372,
a[4,3]=1053/1372,
a[5,1]=13339/49152,
a[5,2]=-4617/16384,
a[5,3]=5427/53248,
a[5,4]=95207/159744,
a[6,1]=-6935/57122,
a[6,2]=23085/48334,
a[6,3]=33363360/273642941,
a[6,4]=972160/118442467,
a[6,5]=172687360/610434253,
a[7,1]=611/1891,
a[7,2]=-4617/7564,
a[7,3]=6041007/13176488,
a[7,4]=12708836/22100117,
a[7,5]=-35840000/62461621,
a[7,6]=6597591/7972456,

b[1]=131/1800,
b[2]=0,
b[3]=1121931/3902080,
b[4]=319333/1682928,
b[5]=262144/2477325,
b[6]=4084223/15177600,
b[7]=1891/25200,

b*[1]=1530637/26100360,
b*[2]=0,
b*[3]=46684107/133951090,
b*[4]=176208361/1789061040,
b*[5]=2228989952/10565208225,
b*[6]=5/24,
b*[7]=1891/25200

. Version: 4 Dec 2011, Peter Stone