An 8 stage, order 6 Runge-Kutta scheme with a 9 stage, order 5 embedded scheme

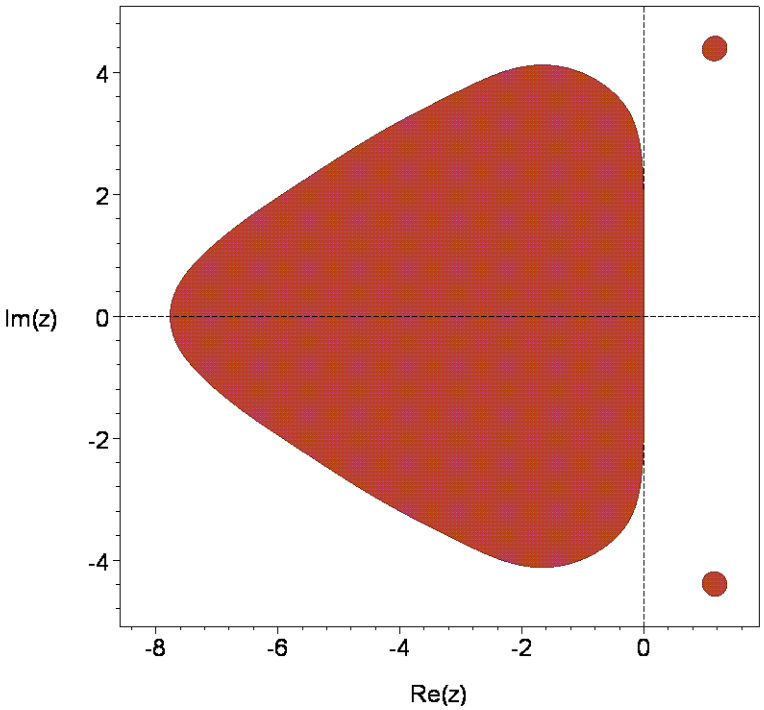
See: On the Optimization of Some Eight-stage Sixth-order Explicit Runge-Kutta Method,
by M. Tanaka, K. Kasuga, S. Yamashita and H. Yazaki,
Journal of the Information Processing Society of Japan, Vol. 34, No. 1 (1993), pages 62 to 74.

The order 6 scheme considered here is that designated as "formula D" in the preceding paper.

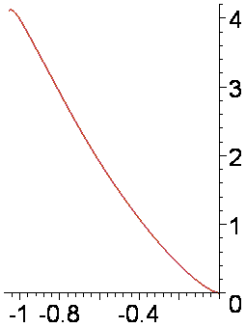
$$c_2 = \frac{1}{250},\ c_3 = \frac{61}{500},\ c_4 = \frac{333}{1000},\ c_5 = \frac{56}{125},\ c_6 = \frac{221}{250},\ c_7 = \frac{31}{40},\ c_9 = 1.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.1575611511 \times 10^{(-3)}$.
Note: The order 6 scheme satisfies 7 of the 48 principal error conditions including the order 7 quadrature condition.
The principal error norm of the order 5 embedded scheme is: $0.1470430320 \times 10^{(-3)}$.
The maximum magnitude of the linking coefficients is: 14.40280909.
The 2-norm of the linking coefficients is: 33.27956217.

The stability region for the order 5 scheme is shown in the following picture.



The stability regions of the order 6 and 5 schemes are "nearly the same".
The real stability intervals of the order 6 and 5 schemes are respectively $[-7.7234, 0]$ and $[-7.7662, 0]$.
The following picture shows the result of distorting the boundary curve of the stability region of the order 6 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis only at the origin.

The coefficients are:
c[2]=1/250,
c[3]=61/500,
c[4]=333/1000,
c[5]=56/125,
c[6]=221/250,
c[7]=31/40,
c[8]=1,
c[9]=1,
a[2,1]=1/250,

a[3,1]=-3477/2000,
a[3,2]=3721/2000,
a[4,1]=5893/960,
a[4,2]=-155333/24000,
a[4,3]=2/3,
a[5,1]=241084612629118721043611274 16/106234180965306940595545374 75,
a[5,2]=-591791939547905415755767172 8/261231592537640017857898462 5,
a[5,3]=232280630406488701857197728/127481017158368328714654449 7,
a[5,4]=54715350804484456909715 20/208985274030112014286318 77,
a[6,1]=643751060321808386212658257155769257511148474857097831643270382 9/
462655382163643505993146289902892129798145865902401535270912000,
a[6,2]=-250831250322389292372115140 53/174154395025093345238598975 0,
a[6,3]=191471592827838703414373042295007003190385413366181230722958611/
201998644533947923598793335502244867385083328952030670310246400,
a[6,4]=-723497411149192638041504597211363403675892243966786723532767/
584075332429951001052530262062711448991365353250616692288000,
a[6,5]=203683958734798947001368949194155698115 19/
122453174578853500453780580140276367360 00,
a[7,1]=-105323552459054450555284772033683179687341534443132552543638772323101240 90077771/
151882847900737440255262782278159042659114250070600779935950570906179610869760,
a[7,2]=110556263584594415870053017499108315/157118342373411699342786117723407 52,
a[7,3]=27883202411953328597139497795565290131372048742640205053329023156822546337651/
663131362709469716828781254768033677324168466826105190970355617617159193886 72,
a[7,4]=-13417868637530706576676771982375909645124556163618041114260383082243504223 9/
191743203036279335802347994313396039656741862830639889775698722784806666240,
a[7,5]=11923653867325940286587110962196582227488711236737921 81/
133998490788434978276482828666342705022977070099070976 0,
a[7,6]=1/16,
a[8,1]=139472860369418680318405579593202554799958275392817771935632674919979204348213061279/
377434266797940532155128323501733508100719588500792149216811825825856916883932938 24,
a[8,2]=-764277739442303538051509575419150250/241488900344982122797870353007980909,
a[8,3]=-737155344446492580657228286298150231948403920352566319118322550863071491042534267 5/
484677931891262920755928965841249174005388337124178994136084513731208068162403379 2,
a[8,4]=880113264267611255492725159071144008154641661854719184251608167727835590018008 5/
207168984264092066098028573350629487028351647669492177921991268566593009121011 2,
a[8,5]=-29836816457137389850003216330123847711815921213507338080455 5/
928286944114663776369892613454549870673362921670343039221 76,
a[8,6]=-240786613447142598518650/148203669033956312374820 3,
a[8,7]=52860332724128242560/47245264125077723987,
a[9,1]=-703354406976784728843895003046219175162208350174561925856136724960707089877205143641859793556 21/
521917305272626136179428588085143385237506707686445983476185050912429396224371881922620209561 6,
a[9,2]=550576025284921509575124738439527563349512 50/382554267281243087076960552094655480130333 9,
a[9,3]=-779807825610508614857513384046891185369530834183000923169185501747700453373240063825744056 15/
394243639208962834576959619954330071955543716106450948343818725123853767229544380653981219 84,
a[9,4]=217753333804594850256418860935700638438169784203810398479593263243995625938633833857415791 9/
387582376874948376545286159615164080595940845312336117877987331598679530792396290947668352,

a[9,5]=-14294659936441121457703497377826200618680137551091558179827346929696858 09/
295236772314145005435682097407317274403116518334589230312060489793912832,
a[9,6]=-16288228679627309596751704537624488187503 5/122605635835516507648831059588971956131 6528,
a[9,7]=61/43,
a[9,8]=0,

b[1]=4783097999/163657290888,
b[2]=0,
b[3]=1043945712500000/4811958427553991,
b[4]=12731480000000/212031396478881,
b[5]=3420427578125/10842916254408,
b[6]=22394286718750/335690566308279,
b[7]=601452300800/2318247773037,
b[8]=11555307017/221477311314,
b[9]=0,

b*[1]=401359714631498171030059910/152670544263820064817403392 87,
b*[2]=0,
b*[3]=187237424719838324134643512 0000/818026229826099930978393320 5623,
b*[4]=215533585653762668869792000/124862096268950345838337185 99,
b*[5]=106439522651274487325284701 25/293188481713142671446782284 86,
b*[6]=499927529297309672174764262 50/454671173972183983205823101 943,
b*[7]=410771061067195844924988684 80/192232681858829974979332840 131,
b*[8]=0,
b*[9]=13/318.

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