# An 8 stage, order 6 Runge-Kutta scheme with a 9 stage, order 5 FSAL embedded scheme

See: A General Family of Explicit Runge-Kutta Pairs of Order 6(5), by S. N. Papakostas, Ch. Tsitouras and G. Papageorgiou,
Siam Journal of Numerical Analysis, Volume 33, Number 3, June 1996, Pages 917-936.

The scheme considered here is constructed in the manner of the schemes in the preceding paper.

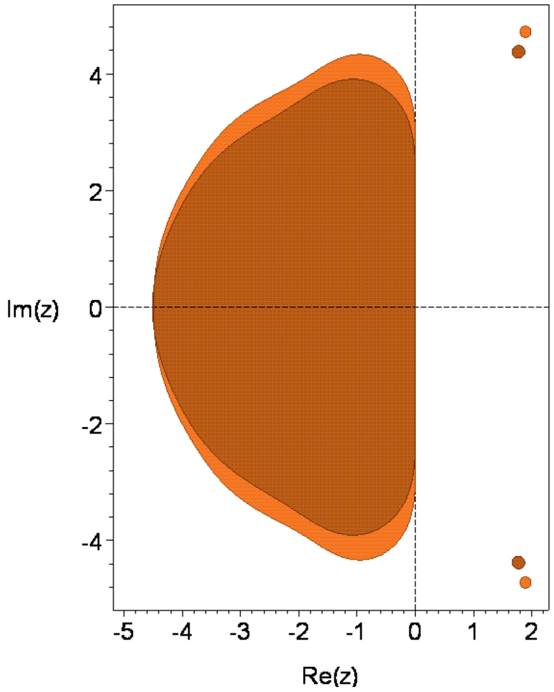
The nodes of the scheme are:

$$c_2 = \frac{3}{34},\ c_3 = \frac{43}{315},\ c_4 = \frac{53}{267},\ c_5 = \frac{63}{118},\ c_6 = \frac{34}{49},\ c_7 = \frac{54}{55},\ c_8 = 1,\ c_9 = 1.$$

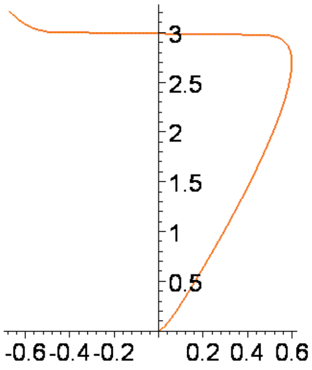
The principal error norm, that is, the 2-norm of the principal error terms is: $0.2189115996 \text{ x } 10^{(-4)}$.

The principal error norm of the order 5 embedded scheme is: $0.6104061815 \text{ x } 10^{(-3)}$.
The maximum magnitude of the linking coefficients is: 26.24493616.
The 2-norm of the linking coefficients is: 44.87341087.

The stability regions for the two schemes are shown in the following picture.



The stability region of the order 5 scheme appears in the darker shade.
The real stability intervals of the order 6 and 5 schemes are respectively $[-4.4943, 0]$ and $[-4.5100, 0]$.
The following picture shows the result of distorting the boundary curve of the stability region of the order 6 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval: $[0, 2.9895]$.

The coefficients correct to 85 digits are:

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c[2]=.8823529411764705882352941176470588235294117647058823529411764705882352941176470588235e-1,
c[3]=.1365079365079365079365079365079365079365079365079365079365079365079365079365079365079,
c[4]=.1985018726591760299625468164794007490636704119850187265917602996254681647940074906367,
c[5]=.5338983050847457627118644067796610169491525423728813559322033898305084745762711864407,
c[6]=.6938775510204081632653061224489795918367346938775510204081632653061224489795918367347,
c[7]=.9818181818181818181818181818181818181818181818181818181818181818181818181818181818182,
c[8]=1.,
c[9]=1.,

a[2,1]=.8823529411764705882352941176470588235294117647058823529411764705882352941176470588235e-1,
a[3,1]=.3091290837322583354329386075417821449567481313513059544805576551608297640043671789704e-1,
a[3,2]=.1055950281347106743932140757537582934408331233728059124884521709918535315360712186109,
a[4,1]=.4693511841993899646732961081714025547945831846082990842371195886021316887816960248408e-1,
a[4,2]=.2047884869883937169330256217434861971664464084797131274154327815239245057353173405250e-1,
a[4,3]=.1310879055403976618019146434879118738675674526762175054265050626128625453423061541001,
a[5,1]=.6610529468964142573155679751230800354882178962801009768024368937090230827801259816874,
a[5,2]=-.8326927613033625821396075435122062034696193289216117411374347564448552898566746827123e-1,
a[5,3]=-2.558029482749860377663969556126937933989200663136458808551129543350876061412973490061,
a[5,4]=2.514144117068528141274226742134739535797097242121400361794639515116846982194786163086,
a[6,1]=-1.376475934957339002017197977702895680246821394453712520271336283174435735647271308545,
a[6,2]=.1396648588193816371241861424783732293412862429481520993026740416134480118599933478393,
a[6,3]=5.989851817426150413300380351017522925325349377199292476185415370854928680033451449846,
a[6,4]=-4.704664514046624210020034571616314420693273217703223794532375878964211564001012051709,
a[6,5]=.6455013237788393248779721782722935381101936858870427597237860149763930567344303993040,
a[7,1]=4.249919408741196657344193827013057236281802277296488374674979644365212676790489232400,
a[7,2]=-.2490013232081823291884046070205078023417012962990368895438776472899894883984296862017,
a[7,3]=-19.03714188512491179672490598904541412712696962764330216256257350647974673480392984482,
a[7,4]=16.83374727132937877836570070989124053728659813363355853296395176242955298847716484417,
a[7,5]=-2.044917524086417745526196116466915269587333388046417017440834642415988877619317158662,
a[7,6]=1.229212234167118253911430357446721243669422082876890980090172571209141253735840794937,
a[8,1]=5.807983376499219054819342247555310058645388871984655586711737681911525357672916425614,
a[8,2]=-.3326730473478644364801748875561755605403788175710541328158077475837884210962323680722,
a[8,3]=-26.24493616233393233299084163597513868371841380575683697846903418255286097706887252854,
a[8,4]=23.14067585086023237998227682721524736225550191745225658679640712093084329615623963707,
a[8,5]=-2.986293389638733849765027984506960029716872509997469651296088377081925188264234299213,
a[8,6]=1.642596383992850507261466113217877237132529140502438431683615511231469930757113172653,
a[8,7]=-.2735301203177132282704067995016038405775479661398984261083000685526399815693003950324e-1,
a[9,1]=.6295101138796060365727419478101582675560408803362973818252786669552677239217216738193e-1,
a[9,2]=0.,
a[9,3]=-.3965362757146183627335908000163578454495572585839194744754143186641842301559724167815e-1,
a[9,4]=.3533353336982079998774102097096349152375236769547686777426945497169294971762278361621,
a[9,5]=.2406310020157196373212265448240265094410225724025982658999309848331102411734092789625,
a[9,6]=.2190543510448688050675202363421454063479552171946893817605624181648680097449482508131,
a[9,7]=.4881800404819019578128826375650925236428913766399756908512312009889051304913400884286,
a[9,8]=-.3244981110571971674629547432202793968800412053672698069894055885329212279625003800700,

b[1]=.6295101138796060365727419478101582675560408803362973818252786669552677239217216738193e-1,
b[2]=0.,
b[3]=-.3965362757146183627335908000163578454495572585839194744754143186641842301559724167815e-1,
b[4]=.3533353336982079998774102097096349152375236769547686777426945497169294971762278361621,
b[5]=.2406310020157196373212265448240265094410225724025982658999309848331102411734092789625,
b[6]=.2190543510448688050675202363421454063479552171946893817605624181648680097449482508131,
b[7]=.4881800404819019578128826375650925236428913766399756908512312009889051304913400884286,
b[8]=-.3244981110571971674629547432202793968800412053672698069894055885329212279625003800700,
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b*[1]=.7056518114016381203705679860640993303386510940906052460663383565961921201837681135745e-1,
b*[2]=0.,
b*[3]=-.1032386029813847826704102370876432286687882704358387791365881385634975738893921804656,
b*[4]=.4253355183355335154152021262151097107333692831814240749369114414003790425934749766531,
b*[5]=.1893895400304267547220947464250201064430713539124450850899248530427129671766523004222,
b*[6]=.2693180925535850705721089682769538280315508495297142661324346520257776191519608356757,
b*[7]=.3825696648610695693178869915035435898208710683425887677646227503744026723428666502966,
b*[8]=-.2272727272727272727272727272727272727272727272727272727272727272727272727272727272727,
b*[9]=-.6666666666666666666666666666666666666666666666666666666666666666666666666666666666667e-2.

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