Sharp-Verner 8 stage, order 6 Runge-Kutta scheme with a 9 stage, order 5 FSAL embedded scheme

See: Completely Imbedded Runge-Kutta Pairs, by P. W. Sharp and J. H. Verner,
SIAM Journal on Numerical Analysis, Vol. 31, No. 4. (Aug., 1994), pages. 1169 to 1190.

The nodes of the scheme are:

$$c_2 = \frac{1}{12},\ c_3 = \frac{2}{15},\ c_4 = \frac{1}{5},\ c_5 = \frac{8}{15},\ c_6 = \frac{2}{3},\ c_7 = \frac{19}{20},\ c_8 = 1,\ c_9 = 1.$$

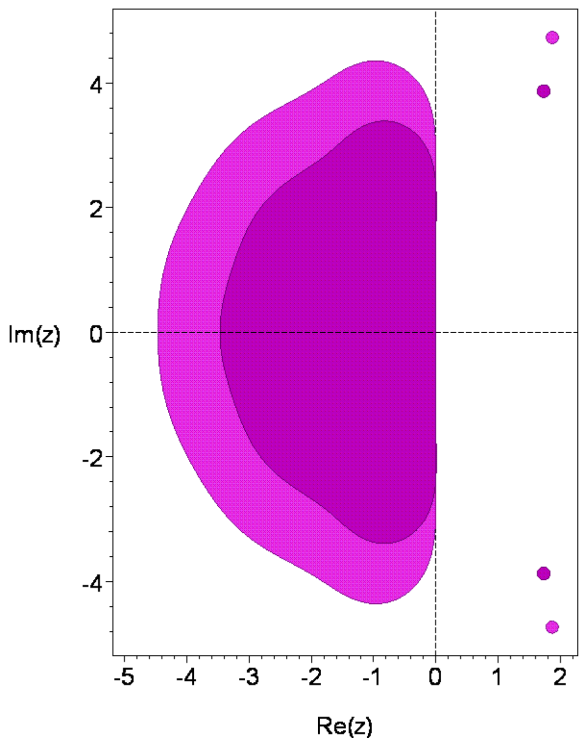
The principal error norm, that is, the 2-norm of the principal error terms is: $0.7945963302 \times 10^{(-4)}$.

The principal error norm of the order 5 embedded scheme is: $0.1924790316 \times 10^{(-2)}$.

The maximum magnitude of the linking coefficients is: $\frac{10956}{2675} \simeq 4.095700935$.

The 2-norm of the linking coefficients is: 9.530433555.

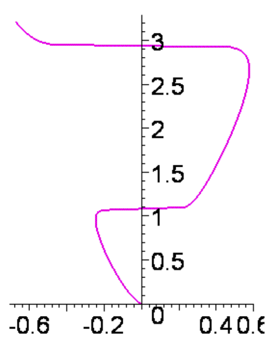
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 5 scheme appears in the darker shade.

The real stability intervals of the order 6 and 5 schemes are respectively $[-4.4708, 0]$ and $[-3.4700, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 6 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval: $[1.0784, 2.9361]$.

The "Butcher" tableau of the scheme is as follows.

$$
\left[\begin{array}{cccccccccc}
\frac{1}{12} & \frac{1}{12} & & & & & & & & \\
\frac{2}{15} & \frac{2}{75} & \frac{8}{75} & & & & & & & \\
\frac{1}{5} & \frac{1}{20} & 0 & \frac{3}{20} & & & & & & \\
\frac{8}{15} & \frac{88}{135} & 0 & -\frac{112}{45} & \frac{64}{27} & & & & & \\
\frac{2}{3} & -\frac{10891}{11556} & 0 & \frac{3880}{963} & -\frac{8456}{2889} & \frac{217}{428} & & & & \\
\frac{19}{20} & \frac{1718911}{4382720} & 0 & -\frac{1000749}{547840} & \frac{819261}{383488} & -\frac{671175}{876544} & \frac{14535}{14336} & & & \\
1 & \frac{85153}{203300} & 0 & -\frac{6783}{2140} & \frac{10956}{2675} & -\frac{38493}{13375} & \frac{1152}{425} & -\frac{7168}{40375} & & \\
1 & \frac{53}{912} & 0 & 0 & \frac{5}{16} & \frac{27}{112} & \frac{27}{136} & \frac{256}{969} & -\frac{25}{336} & \\
b & \frac{53}{912} & 0 & 0 & \frac{5}{16} & \frac{27}{112} & \frac{27}{136} & \frac{256}{969} & -\frac{25}{336} & 0 \\
b* & \frac{617}{10944} & 0 & 0 & \frac{241}{756} & \frac{69}{320} & \frac{435}{1904} & \frac{10304}{43605} & 0 & -\frac{1}{18}
\end{array}\right]
$$

The last-but-one row gives the weights for the order 6 scheme while the last row gives the weights for the embedded order 5 scheme.

The coefficients are:
c[2]=1/12,
c[3]=2/15,
c[4]=1/5,
c[5]=8/15,
c[6]=2/3,
c[7]=19/20,
c[8]=1,
c[9]=1,

a[2,1]=1/12,
a[3,1]=2/75,
a[3,2]=8/75,
a[4,1]=1/20,
a[4,2]=0,
a[4,3]=3/20,
a[5,1]=88/135,
a[5,2]=0,
a[5,3]=-112/45,
a[5,4]=64/27,
a[6,1]=-10891/11556,
a[6,2]=0,
a[6,3]=3880/963,
a[6,4]=-8456/2889,
a[6,5]=217/428,
a[7,1]=1718911/4382720,
a[7,2]=0,
a[7,3]=-1000749/547840,
a[7,4]=819261/383488,
a[7,5]=-671175/876544,
a[7,6]=14535/14336,

a[8,1]=85153/203300,
a[8,2]=0,
a[8,3]=-6783/2140,
a[8,4]=10956/2675,
a[8,5]=-38493/13375,
a[8,6]=1152/425,
a[8,7]=-7168/40375,
a[9,1]=53/912,
a[9,2]=0,
a[9,3]=0,
a[9,4]=5/16,
a[9,5]=27/112,
a[9,6]=27/136,
a[9,7]=256/969,
a[9,8]=-25/336,

b[1]=53/912,
b[2]=0,
b[3]=0,
b[4]=5/16,
b[5]=27/112,
b[6]=27/136,
b[7]=256/969,
b[8]=-25/336,

b*[1]=617/10944,
b*[2]=0,
b*[3]=0,
b*[4]=241/756,
b*[5]=69/320,
b*[6]=435/1904,
b*[7]=10304/43605,
b*[8]=0,
b*[9]=-1/18.

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