John Butcher's 2nd order 6 Runge-Kutta scheme

See: On Runge-Kutta Processes of High Order, by J. C. Butcher,
Journal of the Australian Mathematical Society, Vol. 4, (1964), pages 179 to 194.

The nodes of the scheme are:

$$c_2 = \frac{1}{2} + \frac{\sqrt{5}}{10},\ c_3 = \frac{1}{2} - \frac{\sqrt{5}}{10},\ c_4 = \frac{1}{2} + \frac{\sqrt{5}}{10},\ c_5 = \frac{1}{2} - \frac{\sqrt{5}}{10},\ c_6 = \frac{1}{2} + \frac{\sqrt{5}}{10},\ c_7 = 1.$$

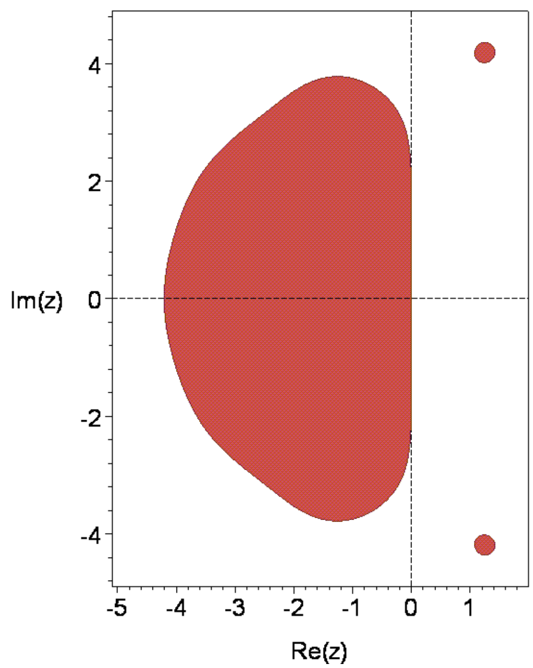
Note: The values:

$$\frac{1}{2} - \frac{\sqrt{5}}{10},\quad \frac{1}{2} + \frac{\sqrt{5}}{10}$$

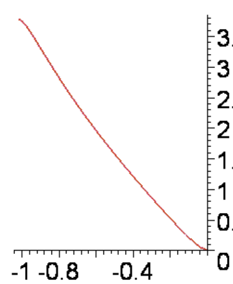
are the zeros of the derivative $P'_3(x) = \frac{d}{dx} P_3(x)$ of the **Legendre polynomial** $P_3(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$. They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

The principal error norm, that is, the 2-norm of the principal error terms is: $0.2372032913 \times 10^{(-2)}$.
The maximum magnitude of the linking coefficients is: $5 + 2\sqrt{5} \simeq 9.472135954$.
The 2-norm of the linking coefficients is: $\frac{\sqrt{99595 + 33915\sqrt{5}}}{30} \simeq 13.96150443$.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-4.2063, 0]$.
The following picture shows the result of distorting the boundary curve of the stability region horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis only at the origin.

The "Butcher" tableau for the scheme is as follows.

$$\begin{bmatrix}
\frac{1}{2}+\frac{\sqrt{5}}{10} & \frac{1}{2}+\frac{\sqrt{5}}{10} & & & & & & \\
\frac{1}{2}-\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & \frac{1}{2}-\frac{\sqrt{5}}{5} & & & & & \\
\frac{1}{2}+\frac{\sqrt{5}}{10} & -\frac{3}{4}-\frac{7\sqrt{5}}{20} & -\frac{1}{4}-\frac{\sqrt{5}}{4} & \frac{3}{2}+\frac{7\sqrt{5}}{10} & & & & \\
\frac{1}{2}-\frac{\sqrt{5}}{10} & \frac{1}{12}+\frac{\sqrt{5}}{60} & 0 & \frac{1}{6} & \frac{1}{4}-\frac{7\sqrt{5}}{60} & & & \\
\frac{1}{2}+\frac{\sqrt{5}}{10} & \frac{1}{12}-\frac{\sqrt{5}}{60} & 0 & \frac{3}{4}+\frac{5\sqrt{5}}{12} & \frac{1}{6} & -\frac{1}{2}-\frac{3\sqrt{5}}{10} & & \\
1 & \frac{1}{6} & 0 & -\frac{55}{12}-\frac{25\sqrt{5}}{12} & -\frac{25}{12}+\frac{7\sqrt{5}}{12} & 5+2\sqrt{5} & \frac{5}{2}-\frac{\sqrt{5}}{2} & \\
 & \frac{1}{12} & 0 & 0 & 0 & \frac{5}{12} & \frac{5}{12} & \frac{1}{12}
\end{bmatrix}$$

The coefficients in exact form are:
c[2]=1/2+1/10*5^(1/2),
c[3]=1/2-1/10*5^(1/2),
c[4]=1/2+1/10*5^(1/2),
c[5]=1/2-1/10*5^(1/2),
c[6]=1/2+1/10*5^(1/2),
c[7]=1,
c[8]=1,

a[2,1]=1/2+1/10*5^(1/2),
a[3,1]=1/10*5^(1/2),
a[3,2]=1/2-1/5*5^(1/2),
a[4,1]=-7/20*5^(1/2)-3/4,
a[4,2]=-1/4*5^(1/2)-1/4,
a[4,3]=3/2+7/10*5^(1/2),
a[5,1]=1/12+1/60*5^(1/2),
a[5,2]=0,a[5,3]=1/6,
a[5,4]=-7/60*5^(1/2)+1/4,
a[6,1]=-1/60*5^(1/2)+1/12,
a[6,2]=0,
a[6,3]=3/4+5/12*5^(1/2),
a[6,4]=1/6,
a[6,5]=-1/2-3/10*5^(1/2),
a[7,1]=1/6,
a[7,2]=0,
a[7,3]=-55/12-25/12*5^(1/2),
a[7,4]=7/12*5^(1/2)-25/12,
a[7,5]=5+2*5^(1/2),a[7,6]=5/2-1/2*5^(1/2),

b[1]=1/12,
b[2]=0,
b[3]=0,
b[4]=0,
b[5]=5/12,
b[6]=5/12,
b[7]=1/12.

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