Bogacki-Shampine combined 7 stage, order 4 and 5 Runge-Kutta scheme with an additional 8 stage, order 4 FSAL embedded scheme

See: An Efficient Runge-Kutta (4,5) Pair by P.Bogacki and L.F.Shampine
Computers and Mathematics with Applications, Vol. 32, No. 6, 1996, pages 15 to 28

The nodes of the scheme are:

$$c_2 = \frac{1}{6},\ c_3 = \frac{2}{9},\ c_4 = \frac{3}{7},\ c_5 = \frac{2}{3},\ c_6 = \frac{3}{4},\ c_7 = 1,\ c_8 = 1.$$

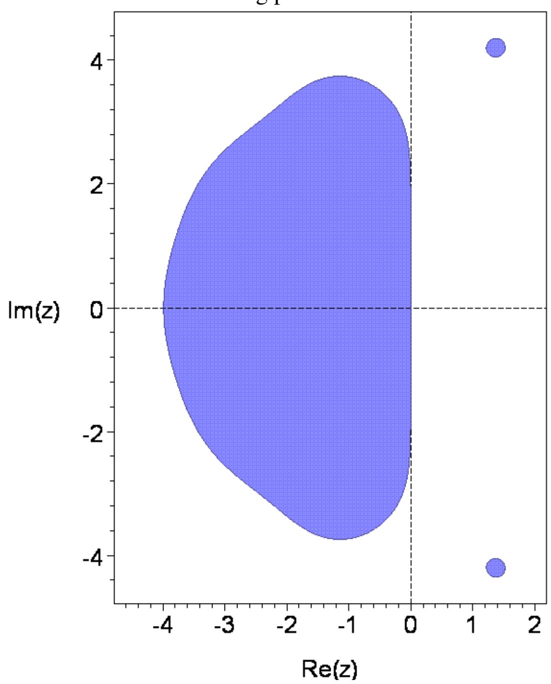
The principal error norm, that is, the 2-norm of the principal error terms is: $0.2216932779 \times 10^{(-4)}$.

The principal error norm of the 7 stage, order 4 embedded scheme is: $0.1059545827 \times 10^{(-3)}$.

The principal error norm of the 8 stage, order 4 embedded scheme is: $0.1061549778 \times 10^{(-3)}$.

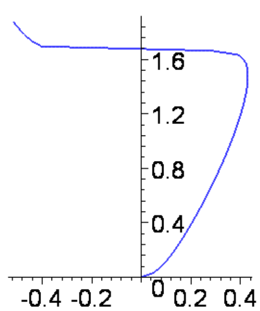
The maximum magnitude of the linking coefficients is: 1.163751542.
The 2-norm of the linking coefficients is: 2.226937100.

The stability region for the order 5 scheme is shown in the following picture.



The real stability interval of the order 5 scheme is $[-3.9879, 0]$.
The stability regions for the two embedded order 4 schemes are very similar in size and shape to that of the order 4 scheme.
The real stability intervals for the 7 stage and 8 stage, order 4 schemes are respectively $[-4.04765, 0]$ and $[-3.9983, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 5 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 1.6643]$.

The Butcher tableau of the scheme is as follows.

$$
\left[\begin{array}{ccccccccc}
\frac{1}{6} & \frac{1}{6} & & & & & & & \\
\frac{2}{9} & \frac{2}{27} & \frac{4}{27} & & & & & & \\
\frac{3}{7} & \frac{183}{1372} & -\frac{162}{343} & \frac{1053}{1372} & & & & & \\
\frac{2}{3} & \frac{68}{297} & -\frac{4}{11} & \frac{42}{143} & \frac{1960}{3861} & & & & \\
\frac{3}{4} & \frac{597}{22528} & \frac{81}{352} & \frac{63099}{585728} & \frac{58653}{366080} & \frac{4617}{20480} & & & \\
1 & \frac{174197}{959244} & -\frac{30942}{79937} & \frac{8152137}{19744439} & \frac{666106}{1039181} & -\frac{29421}{29068} & \frac{482048}{414219} & & \\
1 & \frac{587}{8064} & 0 & \frac{4440339}{15491840} & \frac{24353}{124800} & \frac{387}{44800} & \frac{2152}{5985} & \frac{7267}{94080} & \\
b & \frac{587}{8064} & 0 & \frac{4440339}{15491840} & \frac{24353}{124800} & \frac{387}{44800} & \frac{2152}{5985} & \frac{7267}{94080} & \\
b\hat{} & \frac{6059}{80640} & 0 & \frac{8559189}{30983680} & \frac{26411}{124800} & -\frac{927}{89600} & \frac{443}{1197} & \frac{7267}{94080} & \\
b* & \frac{2479}{34992} & 0 & \frac{123}{416} & \frac{612941}{3411720} & \frac{43}{1440} & \frac{2272}{6561} & \frac{79937}{1113912} & \frac{3293}{556956}
\end{array}\right]
$$

The coefficients are as follows:
c[2]=1/6,
c[3]=2/9,
c[4]=3/7,
c[5]=2/3,
c[6]=3/4,
c[7]=1,
c[8]=1,

a[2,1]=1/6,
a[3,1]=2/27,
a[3,2]=4/27,
a[4,1]=183/1372,
a[4,2]=-162/343,
a[4,3]=1053/1372,
a[5,1]=68/297,
a[5,2]=-4/11,
a[5,3]=42/143,
a[5,4]=1960/3861,
a[6,1]=597/22528,
a[6,2]=81/352,
a[6,3]=63099/585728,
a[6,4]=58653/366080,
a[6,5]=4617/20480,
a[7,1]=174197/959244,
a[7,2]=-30942/79937,
a[7,3]=8152137/19744439,
a[7,4]=666106/1039181,
a[7,5]=-29421/29068,
a[7,6]=482048/414219,

a[8,1]=587/8064,
a[8,2]=0,
a[8,3]=4440339/15491840,
a[8,4]=24353/124800,
a[8,5]=387/44800,
a[8,6]=2152/5985,
a[8,7]=7267/94080,

b[1]=587/8064,
b[2]=0,
b[3]=4440339/15491840,
b[4]=24353/124800,
b[5]=387/44800,
b[6]=2152/5985,
b[7]=7267/94080,

b^[1]=6059/80640,
b^[2]=0,
b^[3]=8559189/30983680,
b^[4]=26411/124800,
b^[5]=-927/89600,
b^[6]=443/1197,
b^[7]=7267/94080,

b*[1]=2479/34992,
b*[2]=0,
b*[3]=123/416,
b*[4]=612941/3411720,
b*[5]=43/1440,
b*[6]=2272/6561,
b*[7]=79937/1113912,
b*[8]=3293/556956.

Version: 20 Dec 2012, Peter Stone