

A modification of Hiroshi Ono's 25 stage, order 12 Runge-Kutta scheme

The scheme considered here is a modification of one due to Hiroshi Ono.

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.

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The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{47}{108}, c_3 = \frac{101}{228}, c_4 = \frac{101}{152}, c_5 = \frac{8255}{76912}, c_6 = \frac{25}{152}, c_9 = \frac{1}{5}, c_{10} = \frac{1}{3}, c_{19} = \frac{1}{3}, c_{20} = \frac{1}{5}, c_{22} = \frac{25}{152}, c_{23} = \frac{101}{228}, c_{24} = \frac{47}{108}, c_{25} = 1,$$

$$b_2 = -\frac{11}{100}, b_3 = -\frac{17}{100}, b_4 = 0, b_5 = 0, b_6 = -\frac{19}{100}, b_7 = -\frac{21}{100}, b_8 = 0, b_9 = -\frac{13}{56}, b_{10} = -\frac{27}{100}, b_{11} = -\frac{29}{100},$$

$$b_{12} = 0, b_{18} = \frac{29}{100}, b_{19} = \frac{27}{100}, b_{20} = \frac{13}{56}, b_{21} = \frac{21}{100}, b_{22} = \frac{19}{100}, b_{23} = \frac{17}{100}, b_{24} = \frac{11}{100}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx}P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

We also have

$$c_7 = \frac{2013}{3445}, c_8 = \frac{17727}{277750},$$

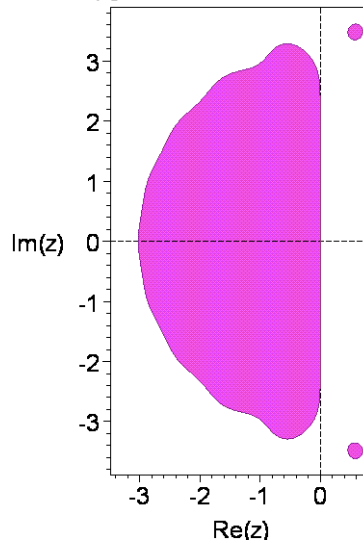
$$c_{11} = \frac{51789075}{64972747} + \frac{333240\sqrt{15}}{64972747} + \frac{3057458\sqrt{495 + 66\sqrt{15}}}{714700217} + \frac{70408\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{714700217},$$

$$c_{12} = \frac{2523614583531}{6408473250305} + \frac{33872310083\sqrt{15}}{1281694650061} - \frac{1080276228947\sqrt{495 + 66\sqrt{15}}}{70493205753355} - \frac{48827895154\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{211479617260065}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.3152572305 \times 10^{(-7)}$.

The maximum magnitude of the linking coefficients is: 212.1164197.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.0248, 0]$.

a[7,2]=0.,
a[7,3]=0.,
a[7,4]=.7610651429965941990931946190296373660533922130208900643092560661920652146268815424885e-1,
a[7,5]=-2.135538596825204678401302463372339475781988628449683891486081018353295131152366999963,
a[7,6]=2.119016745189825526524339470866652730733632823644186992594087231585247087949150559862,
a[8,1]=.3572122856624484555412656871410309454657254402954350529356994466234212462699695879099e-1,
a[8,2]=0.,
a[8,3]=0.,
a[8,4]=0.,
a[8,5]=.4596205641509305161863483311675360886293026632692961086374314175410528945129931429031e-1,
a[8,6]=-1.800016957713219812376781513215261066354472582005125759344515744644655947135707070493e-1,
a[8,7]=.1404669540301245333646491248782654898654978218139650184903068535815036288843799818693e-3,
a[9,1]=.1888176809184108413680419706237505704281864015973418413153299364663524364341955734913e-1,
a[9,2]=0.,
a[9,3]=0.,
a[9,4]=0.,
a[9,5]=0.,
a[9,6]=.8381198329740939383758467268684128106638290469173528975872944409244863365387981657651e-1,
a[9,7]=.9031585241436450659339626877786556727134244569192456020771509670971435444573825121328e-5,
a[9,8]=.9729721702550808557495179062390587533407132090396133365371679075124515126725605224924e-1,
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a[10,2]=0.,
a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=-0.6005539308646711394173617417282635784603029781694346062080009050806164919590318269156,
a[10,7]=.1585222658367901823092810289995660365558277114283363946284113044047141269675809741481e-2,
a[10,8]=.3026658851734428583799288358366834830670493200050903757742452562250879940584583879948,
a[10,9]=.6704368334008921764176894190107687125274815661799611686408713261126274393811928758984,
a[11,1]=6.344326927733666873696263058980778771328558370914152084380122688263116113189631874466,
a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
a[11,5]=0.,
a[11,6]=0.,
a[11,7]=1.975263319684766813850955500728188106557880392559127720135666889111189136253813975729,
a[11,8]=-13.82822337504897849061527131653164721900473695395858210726164571154383357718661916508,
a[11,9]=14.82423926991066347292103944504014293055447712090877566105421769783705551827529470930,
a[11,10]=-8.370994536873562320168249116193728530299734845668945403444620801647194262518201483006,
a[12,1]=-0.9910783781470375735018424706990342484478259963427121190616343432259493286903501070750e-1,
a[12,2]=0.,
a[12,3]=0.,
a[12,4]=0.,
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a[12,6]=-1.046319958132641377892211373672215159710875883372049858922517534871447317958500644882,
a[12,7]=-0.4662578801256029967826301701912208605727776822301916928051722653634090636623902615221e-3,
a[12,8]=.4243518408197806737520879584209906999975132950474577957967837636354467847566731274758,
a[12,9]=.8350767448298191909277817680995907515365477233544915021061336714983604939879298121048,
a[12,10]=-0.62043795620437956204379562043795620437956204379562043795620437956204379562044e-1,
a[12,11]=.3051106025934401220442410373760488176964149504195270785659801678108314263920671243326e-3,
a[13,1]=.1731635805893703684448829089734942070445562382116745420114094326722705203174775140332e-1,
a[13,2]=0.,
a[13,3]=0.,
a[13,4]=0.,
a[13,5]=0.,
a[13,6]=0.,
a[13,7]=0.,
a[13,8]=0.,

a[13,9]=.8690027159880925811795093256330430220184719404902985010762237927304116502630885425916e-3,
a[13,10]=-.9198044746158460357290146202828770811642343120551843217420771155968903404182588708338e-4,
a[13,11]=.1833594777349928706130754092439075395240645268074916590065144720493485758190652981058e-6,
a[13,12]=.6679448817377525524901838117990401028051762116902291244289142812068791591710992924480e-1,
a[14,1]=.1497702285787817250603134142375551527881374815295532946701659971640303704469333256722e-1,
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a[14,3]=0.,
a[14,4]=0.,
a[14,5]=0.,
a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,
a[14,9]=.1299053198125687130806056561360945267563487713700848629960379786258472215118648828622,
a[14,10]=.4444252090193421415880477613428889941728587016143899156135432416265141679149665438354e-2,
a[14,11]=-.2185380330434085597487818850115824768660155325792369439283645364044817815712239904765e-5,
a[14,12]=.6235770138025566019351747531432668514647398310437277172705875380249580436468314885283e-1,
a[14,13]=.5389349250407735998767659637686133399860483467584655047185578940287507515886082812093e-1,
a[15,1]=.1956886038861791180930007029621719896471785620210633245797118818081004119873004962774,
a[15,2]=0.,
a[15,3]=0.,
a[15,4]=0.,
a[15,5]=0.,
a[15,6]=0.,
a[15,7]=0.,
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a[15,11]=.4607633555393391875625052489818876130584589587566695260680043303699764560744838206512e-3,
a[15,12]=-.4833419929869440350491944569648386990066349467457051943375195864364996227787596872297,
a[15,13]=.2837255261146423361717978265203569959020643630988946562818139270625464117810614651744,
a[15,14]=-1.604879577498682731680210867877554840351555444499826924680761144749197624460666828087,
a[16,1]=-0.6628581660952109109950216641584960475848453023977418371554567406302394052049642993363,
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a[16,4]=0.,
a[16,5]=0.,
a[16,6]=0.,
a[16,7]=0.,
a[16,8]=0.,
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a[16,14]=9.736973734199539440165209948098457305642565726332220268712929325968631361580234056276,
a[16,15]=1.047923362573352907746375340805459350884588027111516805638308114257144242834404582751,
a[17,1]=9.451896878619703179615895147719645120861585484250220485534769741441232402797149004402,
a[17,2]=0.,
a[17,3]=0.,
a[17,4]=0.,
a[17,5]=0.,
a[17,6]=0.,
a[17,7]=0.,
a[17,8]=0.,
a[17,9]=74.07837676951819392708315469048815926453257791083353986411194358002277837032529121349,
a[17,10]=80.08971633421252928663933464653279322909492698803384552922422585968931770071180510457,
a[17,11]=-1.241702484260160323471778040612055855620194337087768444994146945534716737633295008233,
a[17,12]=-32.04108125365225923900659816438329678215226279318337501510105142207810131441066638158,

a[17,13]=15.51919421000708125838831543558938623465692911684926717234376165413330242668122537342,
a[17,14]=-136.4444237346563024309541869705527862321862115141467999073204884062451779712319267938,
a[17,15]=-11.36109896858298168444633695622136642878717316645309025294277498926997344146696251530,
a[17,16]=1.746701961099335199963616081872403749335232589961167014444435282876535760443759733215,
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a[18,2]=0.,
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a[18,4]=0.,
a[18,5]=0.,
a[18,6]=0.,
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a[18,9]=-26.72676722061492102175215840759265382960786498230207636418025793583833107088763635551,
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a[18,11]=.7179812487812968675250181644918743516316099586081748670146732996043783143974128607587e-1,
a[18,12]=18.05559723664965139501142471462605880255358278669785143935434349294122071206578011157,
a[18,13]=-8.765163819232982113792324392383185125858432915348234208512916467037776994687248464210,
a[18,14]=76.87522358555575651988028075659565616302805855797883225625477685501643635288672453023,
a[18,15]=6.541007260506904941709851688460080123863085658071943624909334991972860537373864701434,
a[18,16]=-0.8461837296739539683436767973109554382501241857205608248520628152174372445350689400749,
a[18,17]=.3096334815052354314802658810823658907325235844531318754050068324709258105543338930304e-1,
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a[19,3]=0.,
a[19,4]=0.,
a[19,5]=0.,
a[19,6]=-0.6005539308646711394173617417282635784603029781694346062080009050806164919590318269156,
a[19,7]=.1585222658367901823092810289995660365558277114283363946284113044047141269675809741481e-2,
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a[19,9]=.4285248952296136656501504776128834912424841890282487604085652387338305228046948965999,
a[19,10]=-0.2494936049956423479331713062821820760626388763661853215936426940474748269791918051828e-1,
a[19,11]=.2011342149874705950638155217740310555133571768677965217932407885434178872356469888792e-1,
a[19,12]=.1265809640150575117740069284741749647793461966567090378701137188708013785394651655531,
a[19,13]=-0.7625505511021937468044360160133448561674800889340550320120549846220783051800508863794e-2,
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a[19,15]=-0.2745492489167708483879226183906024865124767290578864277433486220883760763766017543321e-1,
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a[19,17]=.1680830476266175444138000910160343423421707378563384687234091879582812751321589095322e-4,
a[19,18]=-0.2135549195295375067495229039525877007339581782873806009604806379884885927885993014667e-1,
a[20,1]=.491872277774214635454694806095471334564471116346374258920200467475606642975663633169e-1,
a[20,2]=0.,
a[20,3]=0.,
a[20,4]=0.,
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a[20,8]=.9729721702550808557495179062390587533407132090396133365371679075124515126725605224924e-1,
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a[20,12]=-0.1273357855008010164642613326804825893357151796749893640146110897003842493364412495521,
a[20,13]=.1195596306045770335798408097988199315656849089154404333183111529360783290389646594900,
a[20,14]=.1268882175226586102719033232628398791540785498489425981873111782477341389728096676737,
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a[20,16]=-0.3909856506984242164691157929655038995858933888334921275260056794557284704385445582377e-2,
a[20,17]=-0.9117410119332842109125953032188840693633119317454401715027062207348542639139803659651e-5,
a[20,18]=.1126093151077456154585623418197514536662349793737574398626301058923382946852045056475e-1,
a[20,19]=-0.3209868434922071245903287586373535845929558438862699119277778588606558307508436673461,

a[21,1]=.5247404461891304721708365639844356354386624470737997647397329116437616959059854701778,
a[21,2]=0.,
a[21,3]=0.,
a[21,4]=.7610651429965941990931946190296373660533922130208900643092560661920652146268815424885e-1,
a[21,5]=-2.135538596825204678401302463372339475781988628449683891486081018353295131152366999963,
a[21,6]=2.119016745189825526524339470866652730733632823644186992594087231585247087949150559862,
a[21,7]=0.,
a[21,8]=0.,
a[21,9]=-6068669292751351984511141135476525247489731841233590933089036540462466843434512014723,
a[21,10]=-6975023816048118989159659127206041759560640169551283418432821453966450135301718130674,
a[21,11]=-2539552135383387231958801508125376392283214913791972115937570999107334296436460657368e-1,
a[21,12]=0.,
a[21,13]=0.,
a[21,14]=0.,
a[21,15]=0.,
a[21,16]=0.,
a[21,17]=0.,
a[21,18]=.2539552135383387231958801508125376392283214913791972115937570999107334296436460657368e-1,
a[21,19]=.6975023816048118989159659127206041759560640169551283418432821453966450135301718130674,
a[21,20]=.6068669292751351984511141135476525247489731841233590933089036540462466843434512014723,
a[22,1]=.3850951504524952575244726520324404860898823844950484490138343541601000605579674908780e-1,
a[22,2]=0.,
a[22,3]=0.,
a[22,4]=.9889604363651382462812798900870548282288802143553903392828423536490053424426345577709e-4,
a[22,5]=.1258652731216402762123982910182735616976625577395859318541619645591514091994326716669,
a[22,6]=0.,
a[22,7]=.1148546657824708136802241649776446744025406100582655444423341878411637297467351936831,
a[22,8]=0.,
a[22,9]=-1299246462925958527077352114079744607750386324042347411360003531859939834460301561195,
a[22,10]=-3664591598580916638621456622089859363144583004356560608229075984337239720284817960193,
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a[22,12]=0.,
a[22,13]=0.,
a[22,14]=0.,
a[22,15]=0.,
a[22,16]=0.,
a[22,17]=0.,
a[22,18]=0.,
a[22,19]=.3664591598580916638621456622089859363144583004356560608229075984337239720284817960193,
a[22,20]=.1299246462925958527077352114079744607750386324042347411360003531859939834460301561195,
a[22,21]=-1148546657824708136802241649776446744025406100582655444423341878411637297467351936831,
a[23,1]=.2175227402212137286104398734798923400326123258875071216675507357419304139407870179368,
a[23,2]=.2254597159191371485825425826604585371603701302528437555254317203984204632521954382036,
a[23,3]=0.,
a[23,4]=0.,
a[23,5]=0.,
a[23,6]=-7003676470588235294117647058823529411764705882352941176470588235294117647058823529412,
a[23,7]=-3841432262252079340469853296205751191707035527677198880717241749265840367609120672261,
a[23,8]=0.,
a[23,9]=0.,
a[23,10]=0.,
a[23,11]=0.,
a[23,12]=0.,
a[23,13]=0.,
a[23,14]=0.,
a[23,15]=0.,
a[23,16]=0.,
a[23,17]=0.,

