

A modification of Hiroshi Ono's 25 stage, order 12 Runge-Kutta scheme

The scheme considered here is a modification of one due to Hiroshi Ono.

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.

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The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{1}{4}, c_3 = \frac{4}{9}, c_4 = \frac{2}{3}, c_5 = \frac{13}{120}, c_6 = \frac{1}{6}, c_9 = \frac{1}{5}, c_{10} = \frac{1}{3}, c_{19} = \frac{1}{3}, c_{20} = \frac{1}{5}, c_{22} = \frac{1}{6}, c_{23} = \frac{4}{9}, c_{24} = \frac{1}{4}, c_{25} = 1,$$

$$b_2 = -\frac{11}{100}, b_3 = -\frac{17}{100}, b_4 = 0, b_5 = 0, b_6 = -\frac{19}{100}, b_7 = -\frac{21}{100}, b_8 = 0, b_9 = -\frac{13}{56}, b_{10} = -\frac{27}{100}, b_{11} = -\frac{29}{100},$$

$$b_{12} = 0, b_{18} = \frac{29}{100}, b_{19} = \frac{27}{100}, b_{20} = \frac{13}{56}, b_{21} = \frac{21}{100}, b_{22} = \frac{19}{100}, b_{23} = \frac{17}{100}, b_{24} = \frac{11}{100}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx}P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

We also have

$$c_7 = \frac{79}{135}, c_8 = \frac{709}{10500},$$

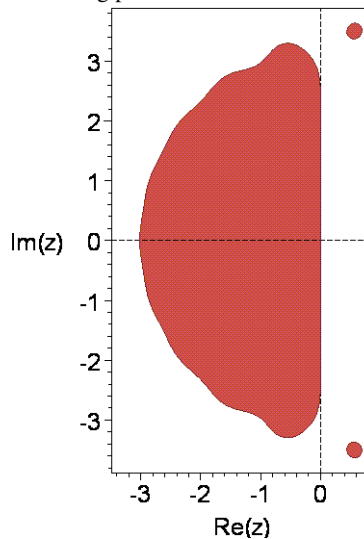
$$c_{11} = \frac{51789075}{64972747} + \frac{333240\sqrt{15}}{64972747} + \frac{3057458\sqrt{495 + 66\sqrt{15}}}{714700217} + \frac{70408\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{714700217},$$

$$c_{12} = \frac{2523614583531}{6408473250305} + \frac{33872310083\sqrt{15}}{1281694650061} - \frac{1080276228947\sqrt{495 + 66\sqrt{15}}}{70493205753355} - \frac{48827895154\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{211479617260065}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.3723238943 \times 10^{(-7)}$.

The maximum magnitude of the linking coefficients is: 190.7463307.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.0193, 0]$.

a[7,2]=0.,
a[7,3]=0.,
a[7,4]=.7543245465503858040489014562406177717822511685842494078406423402976986226718361654108e-1,
a[7,5]=-2.075547971680477873791142836860749355886837204528583027037257940355620678287181163745,
a[7,6]=2.074701465852814732558674031010565075494202151723870973985285827992732428031925059817,
a[8,1]=.3676892528603925219048664448524211597614557351797636700024418222004670689026890820447e-1,
a[8,2]=0.,
a[8,3]=0.,
a[8,4]=0.,
a[8,5]=.4990421592006925486260552960107757259429499555984070931321874496846090397321073859571e-1,
a[8,6]=-1930463579861287136353203957778747781438400759996112964824379316128697839120856079129e-1,
a[8,7]=.1553041163138881199636750152773130534672480459535771443046754965888913372527235149185e-3,
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a[9,4]=0.,
a[9,5]=0.,
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a[9,8]=.9974907654907712890966184600490005170073988116014004264165587607592902930792610781210e-1,
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a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=-6436081333941628246797714549110840967717864507030812404304536614347636764815617688536,
a[10,7]=.1537717626809222300334752193656044639866567030541528516754813628383191013261883003503e-2,
a[10,8]=.3004530256573834493677742186357887570531778112266912246408744303615696525403823773529,
a[10,9]=.7081799766918386616445566840965834495669146999125431907603007531116805020471447502361,
a[11,1]=5.940512035627220006057067930851135219656791728697945883200194895040343032390095577622,
a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
a[11,5]=0.,
a[11,6]=0.,
a[11,7]=1.962055480391860623022348248133501763323649793356240465634479033599078618610316304517,
a[11,8]=-13.64213386305695955284608890764824855130586422219060958260554419417201101166023190424,
a[11,9]=15.06683044587338301779467734916899621382077320594120734305207687034839214126066753426,
a[11,10]=-8.382652493428947744343267048481650586358906421050256154417465842795469852586927600745,
a[12,1]=-8523320090528975844306284116251049341633937016486428835043933336539857150832028583542e-1,
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a[12,5]=0.,
a[12,6]=-1.098569475868715118829812405196151508223952245846956518600662735896710698474868206964,
a[12,7]=-2301722662162744192759150008398127589656586202601802242793932539927412114755307614992e-3,
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a[12,9]=.8876747343695383504499339930841161073464654726641392329690577416246525783568154612140,
a[12,10]=-6289308176100628930817610062893081761006289308176100628930817610062893081761006289308e-1,
a[12,11]=.2792906018712470325373551180002792906018712470325373551180002792906018712470325373551e-3,
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a[13,4]=0.,
a[13,5]=0.,
a[13,6]=0.,
a[13,7]=0.,
a[13,8]=0.,

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a[13,12]=.6679448817377525524901838117990401028051762116902291244289142812068791591710992924480e-1,
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a[14,4]=0.,
a[14,5]=0.,
a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,
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a[14,11]=-.2185380330434085597487818850115824768660155325792369439283645364044817815712239904765e-5,
a[14,12]=.6235770138025566019351747531432668514647398310437277172705875380249580436468314885283e-1,
a[14,13]=.5389349250407735998767659637686133399860483467584655047185578940287507515886082812093e-1,
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a[15,2]=0.,
a[15,3]=0.,
a[15,4]=0.,
a[15,5]=0.,
a[15,6]=0.,
a[15,7]=0.,
a[15,8]=0.,
a[15,9]=1.131705271097428384848259565458044806159353499969331033685458199974824782985315052109,
a[15,10]=.9752663095062898395126139112444816878840152983288853994420604724557591075669232911654,
a[15,11]=.4608796071658958621662391837720327208802327875251834968397345689562589728321333109276e-3,
a[15,12]=-.4852504785332754120709226849433061155990904451582887776294627496994638949105402658215,
a[15,13]=.2858130144624781913379914614162663592935944873020252364609403609188253066146522744959,
a[15,14]=-1.604037876937008259787769885270029435657247097118160214013624556204287480529158597080,
a[16,1]=-0.6109949986287453575309687363729951204037205548100073004743825332314841021236197076812,
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a[16,4]=0.,
a[16,5]=0.,
a[16,6]=0.,
a[16,7]=0.,
a[16,8]=0.,
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a[16,13]=-0.9010915566518024177306418534803759366452867538611049770844158274946846251894973533108,
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a[16,15]=1.023963386517726615974938828712185122866455775709148110245868030218047444885181014611,
a[17,1]=8.516255017869727804522501373003773464660059428763862049049342654476529602025600636532,
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a[17,3]=0.,
a[17,4]=0.,
a[17,5]=0.,
a[17,6]=0.,
a[17,7]=0.,
a[17,8]=0.,
a[17,9]=64.68694746048049302572378727698903533812720953874635165580140508008270857352976575295,
a[17,10]=74.16040605818612945613741812973898521087566764052253903924113051407170318558032887225,
a[17,11]=-1.284224513797453333522083546040060198717544515995961390842264690095562649421470483175,
a[17,12]=-30.52371644368934566936045055595589350203331631109468024677617461888399083798749057490,

a[17,13]=16.82485005548533679904679783839199464721260540068357758907799768438780172547709049255,
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a[17,15]=-10.93192287327478137255205044839870453099517147509932616374906907265036931593174181607,
a[17,16]=1.746701961099335199963616081872403749335232589961167014444435282876535760443759733215,
a[18,1]=1.193377205149009778765199259239075262530149825442799862260349042036462597778349447516,
a[18,2]=0.,
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a[18,5]=0.,
a[18,6]=0.,
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a[18,11]=.7424225834311905032622105807301803404296132320209565206538803299362644212495968980587e-1,
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a[18,14]=69.39918769510392366654746010486216852516161308098182112481592844877572960010657049176,
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a[18,16]=-0.8461837296739539683436767973109554382501241857205608248520628152174372445350689400749,
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a[19,3]=0.,
a[19,4]=0.,
a[19,5]=0.,
a[19,6]=-0.6436081333941628246797714549110840967717864507030812404304536614347636764815617688536,
a[19,7]=.1537717626809222300334752193656044639866567030541528516754813628383191013261883003503e-2,
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a[20,1]=.5454908705059708185473223746577815076037041570748372887925259242883324801498288967757e-1,
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a[20,13]=.1151395839604398867815003854362062247765862709141976243746590116338986773309243127535,
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a[20,16]=-0.3909856506984242164691157929655038995858933888334921275260056794557284704385445582377e-2,
a[20,17]=-0.9117410119332842109125953032188840693633119317454401715027062207348542639139803659651e-5,
a[20,18]=.1121029070910995545199917388600394550626832254247437200910915070169440797714588293937e-1,
a[20,19]=-0.3344481605351170568561872909698996655518394648829431438127090301003344481605351170569,

a[21,1]=.5105992363578097460127638454113076883995951211314722974530930635183035731732576725718,
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a[21,4]=.7543245465503858040489014562406177717822511685842494078406423402976986226718361654108e-1,
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a[21,7]=0.,
a[21,8]=0.,
a[21,9]=-.6056348202998013364637890554451869243272530251038468484739028354704713743904641502619,
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a[21,11]=-.2530464169447410633467649077644850846834331528772883593493989655560821683241998040578e-1,
a[21,12]=0.,
a[21,13]=0.,
a[21,14]=0.,
a[21,15]=0.,
a[21,16]=0.,
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a[21,19]=.6962774898931662522054431030410802850752281852155429905113849488059348994873647212459,
a[21,20]=.6056348202998013364637890554451869243272530251038468484739028354704713743904641502619,
a[22,1]=.3899572649572649572649572649572649572649572649572649572649572649572649572649573e-1,
a[22,2]=0.,
a[22,3]=0.,
a[22,4]=.1036484245439469320066334991708126036484245439469320066334991708126036484245439469320e-3,
a[22,5]=.1275672917463962240081643066717693583365225156269932389335374410001275672917463962240,
a[22,6]=0.,
a[22,7]=.1146176189444805429210887300555916540321998411667027651433109522778138762544220633889,
a[22,8]=0.,
a[22,9]=-.1316367023754075454122030740568234746157428970656730321378667908709827666511411271542,
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a[22,12]=0.,
a[22,13]=0.,
a[22,14]=0.,
a[22,15]=0.,
a[22,16]=0.,
a[22,17]=0.,
a[22,18]=0.,
a[22,19]=.3845365626455519329296693060083837913367489520260829063809967396367023754075454122031,
a[22,20]=.1316367023754075454122030740568234746157428970656730321378667908709827666511411271542,
a[22,21]=-.1146176189444805429210887300555916540321998411667027651433109522778138762544220633889,
a[23,1]=.4938271604938271604938271604938271604938271604938271604938271604938271604938271604938e-1,
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a[23,3]=0.,
a[23,4]=0.,
a[23,5]=0.,
a[23,6]=-.6985294117647058823529411764705882352941176470588235294117647058823529411764705882353,
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a[23,8]=0.,
a[23,9]=0.,
a[23,10]=0.,
a[23,11]=0.,
a[23,12]=0.,
a[23,13]=0.,
a[23,14]=0.,
a[23,15]=0.,
a[23,16]=0.,
a[23,17]=0.,

a[23,18]=0.,
a[23,19]=0.,
a[23,20]=0.,
a[23,21]=.3834558992979908012587751149842653110626966836117162914548535463568143306705398208666,
a[23,22]=.6985294117647058823529411764705882352941176470588235294117647058823529411764705882353,
a[24,1]=.25,
a[24,2]=0.,
a[24,3]=-3179947624392068836513280957725402169846614291058735503179947624392068836513280957725,
a[24,4]=0.,
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a[24,6]=0.,
a[24,7]=0.,
a[24,8]=0.,
a[24,9]=0.,
a[24,10]=0.,
a[24,11]=0.,
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a[24,16]=0.,
a[24,17]=0.,
a[24,18]=0.,
a[24,19]=0.,
a[24,20]=0.,
a[24,21]=0.,
a[24,22]=0.,
a[24,23]=.3179947624392068836513280957725402169846614291058735503179947624392068836513280957725,
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a[25,2]=-3.465,
a[25,3]=-2.497530864197530864197530864197530864197530864197530864197530864197530864197530864198,
a[25,4]=0.,
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a[25,6]=-1.6625,
a[25,7]=-1.835440144855967078189300411522633744855967078189300411522633744855967078189300411523,
a[25,8]=0.,
a[25,9]=97.46352513916303401275956068612342438623474537647956512485332269672990225420949696705,
a[25,10]=113.6558001777249777440450100991518115934070211549269449208208934361619801936764163619,
a[25,11]=-7981167931425572527946707433039421894591043836927747846033502221334839698586183800463,
a[25,12]=-47.54514209861785588208647022162854245033141471110391674396201025649878929561875114933,
a[25,13]=27.01907235256904785559950590961079758017863246649608801708370410498327319562326004765,
a[25,14]=-190.7463307284624556328396816905756117308149301787466966133516762583373098548084289408,
a[25,15]=-17.18558882190315590192721142315864658147129103136539262024710060553179087040443533161,
a[25,16]=2.509716434086569985363077956674238286724267256774032757480361264001771307725305753519,
a[25,17]=1.162475886937088360535666476065014131326443901611275096836079807311057386735832665899,
a[25,18]=.5839488303468939449260909132929399737902477241002934521537006256042283009612379572628,
a[25,19]=1.581,
a[25,20]=1.33584,
a[25,21]=1.835440144855967078189300411522633744855967078189300411522633744855967078189300411523,
a[25,22]=1.6625,
a[25,23]=2.497530864197530864197530864197530864197530864197530864197530864197530864197530864198,
a[25,24]=3.465,

b[1]=.2380952380952380952380952380952380952380952380952380952380952380952380952380952380952380952380952e-1,
b[2]=-11,
b[3]=-17,
b[4]=0.,
b[5]=0.,

