

A modification of Hiroshi Ono's 25 stage, order 12 Runge-Kutta scheme

The scheme considered here is a modification of one due to Hiroshi Ono.

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.

Transactions of the Japan Society for Industrial and applied Mathematics, Vol. 6, No. 3, (2006) pages 177 to 186.

The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{39}{88}, c_3 = \frac{101}{228}, c_4 = \frac{101}{152}, c_5 = \frac{8255}{76912}, c_6 = \frac{25}{152}, c_9 = \frac{1}{5}, c_{10} = \frac{1}{3}, c_{19} = \frac{1}{3}, c_{20} = \frac{1}{5}, c_{22} = \frac{25}{152}, c_{23} = \frac{101}{228}, c_{24} = \frac{39}{88}, c_{25} = 1,$$

$$b_2 = -\frac{11}{100}, b_3 = -\frac{17}{100}, b_4 = 0, b_5 = 0, b_6 = -\frac{19}{100}, b_7 = -\frac{21}{100}, b_8 = 0, b_9 = -\frac{13}{56}, b_{10} = -\frac{27}{100}, b_{11} = -\frac{29}{100},$$

$$b_{12} = 0, b_{18} = \frac{29}{100}, b_{19} = \frac{27}{100}, b_{20} = \frac{13}{56}, b_{21} = \frac{21}{100}, b_{22} = \frac{19}{100}, b_{23} = \frac{17}{100}, b_{24} = \frac{11}{100}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx}P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

We also have

$$c_7 = \frac{2013}{3445}, c_8 = \frac{17727}{277750},$$

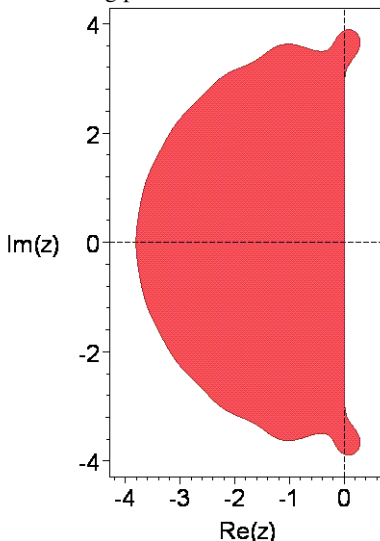
$$c_{11} = \frac{51789075}{64972747} + \frac{333240\sqrt{15}}{64972747} + \frac{3057458\sqrt{495 + 66\sqrt{15}}}{714700217} + \frac{70408\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{714700217},$$

$$c_{12} = \frac{2523614583531}{6408473250305} + \frac{33872310083\sqrt{15}}{1281694650061} - \frac{1080276228947\sqrt{495 + 66\sqrt{15}}}{70493205753355} - \frac{48827895154\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{211479617260065}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.4222219839 \times 10^{(-7)}$.

The maximum magnitude of the linking coefficients is: 97.33866706.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.8084, 0]$.

a[7,4]=.7610651429965941990931946190296373660533922130208900643092560661920652146268815424885e-1,
a[7,5]=-2.135538596825204678401302463372339475781988628449683891486081018353295131152366999963,
a[7,6]=2.119016745189825526524339470866652730733632823644186992594087231585247087949150559862,
a[8,1]=.3572122856624484555412656871410309454657254402954350529356994466234212462699695879099e-1,
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a[8,4]=0.,
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a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=-.6005539308646711394173617417282635784603029781694346062080009050806164919590318269156,
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a[10,9]=.6704368334008921764176894190107687125274815661799611686408713261126274393811928758984,
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a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
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a[11,6]=0.,
a[11,7]=1.975263319684766813850955500728188106557880392559127720135666889111189136253813975729,
a[11,8]=-13.82822337504897849061527131653164721900473695395858210726164571154383357718661916508,
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a[11,10]=-8.370994536873562320168249116193728530299734845668945403444620801647194262518201483006,
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a[15,6]=0.,
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a[16,15]=.9314731355465816846944233097416818433858461568114785937701677417314453177084516021079,
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a[17,11]=-1.438278891922184027856469811705130362411606484021016873421968131077466240414024785286,
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a[17,13]=39.98152490677138860229236700419324273628266379855435510326340337917829130557505135092,
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a[17,15]=-9.275218180677629986716541889092615163371162093764164198259568827927787947256575200423,
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a[18,15]=5.342112094299260796003572810207495392624273510487948724338733165954427246791949416984,
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a[19,4]=0.,
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a[21,20]=0.6125209689888786940888574127421958712528456050934525010105394023758701006572100635978,
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a[22,6]=0.,
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a[22,11]=0.,
a[22,12]=0.,
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a[23,3]=0.,
a[23,4]=0.,
a[23,5]=0.,
a[23,6]=-0.7003676470588235294117647058823529411764705882352941176470588235294117647058823529412,
a[23,7]=-0.3841432262252079340469853296205751191707035527677198880717241749265840367609120672261,
a[23,8]=0.,
a[23,9]=0.,
a[23,10]=0.,
a[23,11]=0.,
a[23,12]=0.,
a[23,13]=0.,
a[23,14]=0.,
a[23,15]=0.,
a[23,16]=0.,
a[23,17]=0.,
a[23,18]=0.,
a[23,19]=0.,

b[8]=0.,
b[9]=- .2321428571428571428571428571428571428571428571428571428571428571428571428571428571429,
b[10]=- .27,
b[11]=- .29,
b[12]=0.,
b[13]=.1384130236807829740053502031450331467488136400899412345912671194817223119377730668077,
b[14]=.2158726906049313117089355111406811389654720741957730511230185948039919737765126474781,
b[15]=.2438095238095238095238095238095238095238095238095238095238095238095238095238095238095,
b[16]=.2158726906049313117089355111406811389654720741957730511230185948039919737765126474781,
b[17]=.1384130236807829740053502031450331467488136400899412345912671194817223119377730668077,
b[18]=.29,
b[19]=.27,
b[20]=.2321428571428571428571428571428571428571428571428571428571428571428571428571428571429,
b[21]=.21,
b[22]=.19,
b[23]=.17,
b[24]=.11,
b[25]=.2380952380952380952380952380952380952380952380952380952380952380952380952380952380952e-1.

Version: 30 Nov 2011, Peter Stone