

Hiroshi Ono's 25 stage, order 12 Runge-Kutta scheme

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.

Transactions of the Japan Society for Industrial and applied Mathematics, Vol. 6, No. 3, (2006) pages 177 to 186.

The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{1}{4}, c_3 = \frac{4}{9}, c_4 = \frac{2}{3}, c_5 = \frac{13}{120}, c_6 = \frac{1}{6}, c_9 = \frac{1}{5}, c_{10} = \frac{1}{3}, c_{19} = \frac{1}{3}, c_{20} = \frac{1}{5}, c_{22} = \frac{1}{6}, c_{23} = \frac{4}{9}, c_{24} = \frac{1}{4}, c_{25} = 1,$$

$$b_2 = -\frac{11}{100}, b_3 = -\frac{17}{100}, b_4 = 0, b_5 = 0, b_6 = -\frac{19}{100}, b_7 = -\frac{21}{100}, b_8 = 0, b_9 = -\frac{23}{100}, b_{10} = -\frac{27}{100}, b_{11} = -\frac{29}{100},$$

$$b_{12} = 0, b_{18} = \frac{29}{100}, b_{19} = \frac{27}{100}, b_{20} = \frac{23}{100}, b_{21} = \frac{21}{100}, b_{22} = \frac{19}{100}, b_{23} = \frac{17}{100}, b_{24} = \frac{11}{100}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx}P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

We also have

$$c_7 = \frac{79}{135}, c_8 = \frac{709}{10500},$$

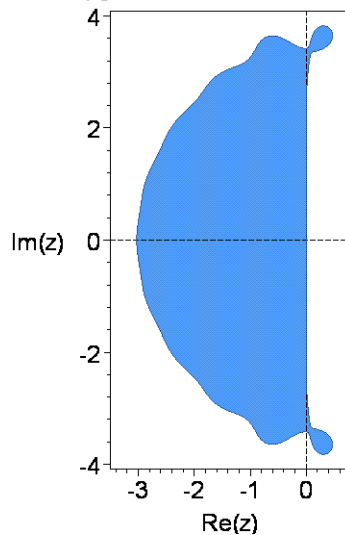
$$c_{11} = \frac{51789075}{64972747} + \frac{333240\sqrt{15}}{64972747} + \frac{3057458\sqrt{495 + 66\sqrt{15}}}{714700217} + \frac{70408\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{714700217},$$

$$c_{12} = \frac{2523614583531}{6408473250305} + \frac{33872310083\sqrt{15}}{1281694650061} - \frac{1080276228947\sqrt{495 + 66\sqrt{15}}}{70493205753355} - \frac{48827895154\sqrt{495 + 66\sqrt{15}}\sqrt{15}}{211479617260065}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.3097360773 \times 10^{(-6)}$.

The maximum magnitude of the linking coefficients is: 73.24196602.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.0281, 0]$.

a[7,2]=0.,
a[7,3]=0.,
a[7,4]=.7543245465503858040489014562406177717822511685842494078406423402976986226718361654108e-1,
a[7,5]=-2.075547971680477873791142836860749355886837204528583027037257940355620678287181163745,
a[7,6]=2.074701465852814732558674031010565075494202151723870973985285827992732428031925059817,
a[8,1]=.3676892528603925219048664448524211597614557351797636700024418222004670689026890820447e-1,
a[8,2]=0.,
a[8,3]=0.,
a[8,4]=0.,
a[8,5]=.4990421592006925486260552960107757259429499555984070931321874496846090397321073859571e-1,
a[8,6]=-1930463579861287136353203957778747781438400759996112964824379316128697839120856079129e-1,
a[8,7]=.1553041163138881199636750152773130534672480459535771443046754965888913372527235149185e-3,
a[9,1]=.2067486743675349484922604488404063487529235328774704968666869007873453428790773240971e-1,
a[9,2]=0.,
a[9,3]=0.,
a[9,4]=0.,
a[9,5]=0.,
a[9,6]=.7957154880008160975236540766621611282548264517609854377598122975695595623677029405014e-1,
a[9,7]=-4507214087766488746701444843200598485120376014363895694204088380480167395865728052023e-5,
a[9,8]=.9974907654907712890966184600490005170073988116014004264165587607592902930792610781210e-1,
a[10,1]=-3322925324853517529956086668161082115483929413336137015414300233353633578589390840556e-1,
a[10,2]=0.,
a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=-6436081333941628246797714549110840967717864507030812404304536614347636764815617688536,
a[10,7]=.1537717626809222300334752193656044639866567030541528516754813628383191013261883003503e-2,
a[10,8]=.3004530256573834493677742186357887570531778112266912246408744303615696525403823773529,
a[10,9]=.7081799766918386616445566840965834495669146999125431907603007531116805020471447502361,
a[11,1]=5.940512035627220006057067930851135219656791728697945883200194895040343032390095577622,
a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
a[11,5]=0.,
a[11,6]=0.,
a[11,7]=1.962055480391860623022348248133501763323649793356240465634479033599078618610316304517,
a[11,8]=-13.64213386305695955284608890764824855130586422219060958260554419417201101166023190424,
a[11,9]=15.06683044587338301779467734916899621382077320594120734305207687034839214126066753426,
a[11,10]=-8.382652493428947744343267048481650586358906421050256154417465842795469852586927600745,
a[12,1]=.2644349979861653392555978654560316035400311240360155334959299612007139374157863919974e-1,
a[12,2]=0.,
a[12,3]=0.,
a[12,4]=0.,
a[12,5]=0.,
a[12,6]=0.,
a[12,7]=0.,
a[12,8]=.2907477949164468491495277367249908074184860785029007469445512693351379740267954217878e-1,
a[12,9]=-4670146655526017247312261556616256909459346454842633293588922375484653326459852610286e-2,
a[12,10]=.9537840586679400528486700806869537239400914426861191427349413827712021873350961548845e-3,
a[12,11]=-6069889118531287692814140320863412762496957479595228817306174863669287380259690136170e-5,
a[13,1]=.1731635805893703684448829089734942070445562382116745420114094326722705203174775140332e-1,
a[13,2]=0.,
a[13,3]=0.,
a[13,4]=0.,
a[13,5]=0.,
a[13,6]=0.,
a[13,7]=0.,
a[13,8]=0.,

a[13,9]=.8690027159880925811795093256330430220184719404902985010762237927304116502630885425916e-3,
a[13,10]=-.9198044746158460357290146202828770811642343120551843217420771155968903404182588708338e-4,
a[13,11]=.1833594777349928706130754092439075395240645268074916590065144720493485758190652981058e-6,
a[13,12]=.6679448817377525524901838117990401028051762116902291244289142812068791591710992924480e-1,
a[14,1]=.1497702285787817250603134142375551527881374815295532946701659971640303704469333256722e-1,
a[14,2]=0.,
a[14,3]=0.,
a[14,4]=0.,
a[14,5]=0.,
a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,
a[14,9]=.1299053198125687130806056561360945267563487713700848629960379786258472215118648828622,
a[14,10]=.4444252090193421415880477613428889941728587016143899156135432416265141679149665438354e-2,
a[14,11]=-.2185380330434085597487818850115824768660155325792369439283645364044817815712239904765e-5,
a[14,12]=.6235770138025566019351747531432668514647398310437277172705875380249580436468314885283e-1,
a[14,13]=.5389349250407735998767659637686133399860483467584655047185578940287507515886082812093e-1,
a[15,1]=1.188479339801401774163727997421987897149748610296498687871610457754515001479951581977,
a[15,2]=0.,
a[15,3]=0.,
a[15,4]=0.,
a[15,5]=0.,
a[15,6]=0.,
a[15,7]=0.,
a[15,8]=0.,
a[15,9]=-2.153577860743923948574230282195458010685777028877789457391059928453625236658492400496,
a[15,10]=.4084894136568439103913845855092578246970391902744397258692599626546701227557648623350,
a[15,11]=.7865354687677762592501346934798315168840512712394797311164536002596101630809266673394e-3,
a[15,12]=-5.831493996616730192196764781137026841656419605282914766077967205018020861054852689274,
a[15,13]=6.133497357760298833907642604705486519224977667578896200018420236452356862070881047498,
a[15,14]=.7538192106733418460489897410022727797535471147396301299786200230098445012436666712925,
a[16,1]=-1.155141455453753782409702338155440396898375536349635780553293168727187831583039150444,
a[16,2]=0.,
a[16,3]=0.,
a[16,4]=0.,
a[16,5]=0.,
a[16,6]=0.,
a[16,7]=0.,
a[16,8]=0.,
a[16,9]=5.230035453923383710410708524533553450562429691771028687028452868714084651714450832431,
a[16,10]=-.6054719632283663895583349987822330203147735874309119730944517939675864870705141356280,
a[16,11]=.9098600291918418987613542477924427728076737025978819058456352939211248672787562673474e-2,
a[16,12]=7.389890569427277711001889579558669404987538661035016661449808071907466968267996694225,
a[16,13]=-8.586861200440706242153541145492405736845705766762479792176742622565505838929692855382,
a[16,14]=-2.289142858952360074158926453709367741105307029478199282759359773927787664228731981389,
a[16,15]=.7420172511679637547821792305236827765889155660251250395984647953087827182153068879117,
a[17,1]=-3878704284298115050812248647831359622498421729413088096991246117344300538467378525660,
a[17,2]=0.,
a[17,3]=0.,
a[17,4]=0.,
a[17,5]=0.,
a[17,6]=0.,
a[17,7]=0.,
a[17,8]=0.,
a[17,9]=-52.79686296396567660865119351124975527608768115963057325690048513861113171732667960572,
a[17,10]=3.185273134611601001666377466850930616911166170460075757653190749060280689442040942426,
a[17,11]=-1.777687036844069342486311680075793729237679916897779073551385167909574116891757259578,
a[17,12]=-24.01291564936409810460023986270697223960990857367094000364464827510022831834721791049,

a[17,13]=44.59765117143356991664568418613870421297485805339864304716728296273494210914975546336,
a[17,14]=34.64254823700930167755101801088405791740501138964826361776800135713319689123911661552,
a[17,15]=-5.881644810570531178309390232014921075961483123099551413638107203551765688978516422159,
a[17,16]=1.746701961099335199963616081872403749335232589961167014444435282876535760443759733215,
a[18,1]=7.233966671341530645490952497192110761349200861822849620574804240945984259787128547890,
a[18,2]=0.,
a[18,3]=0.,
a[18,4]=0.,
a[18,5]=0.,
a[18,6]=0.,
a[18,7]=1.962055480391860623022348248133501763323649793356240465634479033599078618610316304517,
a[18,8]=-13.64213386305695955284608890764824855130586422219060958260554419417201101166023190424,
a[18,9]=43.38367605377242869214585350579838046755997844867070816778334123860217800450037322178,
a[18,10]=-9.834713911815061145761755868590694958877038434912171502071857284257585145054082454689,
a[18,11]=.1029076524695381286377868412846724978000944487949691840993018167527636501337920551920,
a[18,12]=8.468423106151264851550086751522745687779805425203659086774340333971055043511878909430,
a[18,13]=-20.03922776536488408316747608806415623443728880893196401395082287814408336477871062892,
a[18,14]=-19.26671999607368316162814844363603221030906811653440141113230651624038103492726683046,
a[18,15]=3.391598559113951777436829245234173685429846516750495577069567102933678571370358241684,
a[18,16]=-0.8461837296739539683436767973109554382501241857205608248520628152174372445350689400749,
a[18,17]=.3096334815052354314802658810823658907325235844531318754050068324709258105543338930304e-1,
a[19,1]=-0.8082042305712324316381428055181002921423821181387879506081394201551828774972588869862,
a[19,2]=0.,
a[19,3]=0.,
a[19,4]=0.,
a[19,5]=0.,
a[19,6]=-0.6436081333941628246797714549110840967717864507030812404304536614347636764815617688536,
a[19,7]=.1537717626809222300334752193656044639866567030541528516754813628383191013261883003503e-2,
a[19,8]=.3004530256573834493677742186357887570531778112266912246408744303615696525403823773529,
a[19,9]=1.629527090754742423152222936373756585439149301882472540854080995266699866827769539264,
a[19,10]=-0.2429977955477512310051787601585012098624988470344665355076878808184221276200961070016,
a[19,11]=.1937321543128929118688572792462119078051789173168830514027044834102257771758364823616e-1,
a[19,12]=3.652148675291515690405068402145314798647603571421741897364878082465880262359243365600,
a[19,13]=-3.575712494072609116114226433534392208450672236215592934178802498800227318056660241871,
a[19,14]=0.,
a[19,15]=.1424995469019583371886649267486631800285185459043846216158470864873378043488691875332e-1,
a[19,16]=.7783761260627937338705544952370637303437265442198231246348540202794422441986844495010e-2,
a[19,17]=.1680830476266175444138000910160343423421707378563384687234091879582812751321589095322e-4,
a[19,18]=-0.2123426209823757245364666745406479474016549497429582971324756529195024847371745454980e-1,
a[20,1]=.3030500118630462312535985041299218980792680301551577592229328311072204102313787331391,
a[20,2]=0.,
a[20,3]=0.,
a[20,4]=0.,
a[20,5]=0.,
a[20,6]=.7957154880008160975236540766621611282548264517609854377598122975695595623677029405014e-1,
a[20,7]=.4507214087766488746701444843200598485120376014363895694204088380480167395865728052023e-5,
a[20,8]=.9974907654907712890966184600490005170073988116014004264165587607592902930792610781210e-1,
a[20,9]=-0.3171617769987166876020401487839312465336733321856991443597739463291092021830833497646,
a[20,10]=.4097746751846734839576474906854331980839883999186796914837456369263814852323254415473,
a[20,11]=-0.1037579600906669233865265654007540316189053766128282845577718841615974679757246082771e-1,
a[20,12]=-1.317367560379299247049913838592471137800861178079091445393764303876611018306956677528,
a[20,13]=1.279912157519227083039575094951746274090584680488104164669880623922417279542229865347,
a[20,14]=0.,
a[20,15]=0.,
a[20,16]=-0.3909856506984242164691157929655038995858933888334921275260056794557284704385445582377e-2,
a[20,17]=-0.9117410119332842109125953032188840693633119317454401715027062207348542639139803659651e-5,
a[20,18]=.1121029070910995545199917388600394550626832254247437200910915070169440797714588293937e-1,
a[20,19]=-0.3344481605351170568561872909698996655518394648829431438127090301003344481605351170569,

a[21,1]=.5105992363578097460127638454113076883995951211314722974530930635183035731732576725718,
a[21,2]=0.,
a[21,3]=0.,
a[21,4]=.7543245465503858040489014562406177717822511685842494078406423402976986226718361654108e-1,
a[21,5]=-2.075547971680477873791142836860749355886837204528583027037257940355620678287181163745,
a[21,6]=2.074701465852814732558674031010565075494202151723870973985285827992732428031925059817,
a[21,7]=0.,
a[21,8]=0.,
a[21,9]=-.6056348202998013364637890554451869243272530251038468484739028354704713743904641502619,
a[21,10]=-.6962774898931662522054431030410802850752281852155429905113849488059348994873647212459,
a[21,11]=-.2530464169447410633467649077644850846834331528772883593493989655560821683241998040578e-1,
a[21,12]=0.,
a[21,13]=0.,
a[21,14]=0.,
a[21,15]=0.,
a[21,16]=0.,
a[21,17]=0.,
a[21,18]=.2530464169447410633467649077644850846834331528772883593493989655560821683241998040578e-1,
a[21,19]=.6962774898931662522054431030410802850752281852155429905113849488059348994873647212459,
a[21,20]=.6056348202998013364637890554451869243272530251038468484739028354704713743904641502619,
a[22,1]=.3899572649572649572649572649572649572649572649572649572649572649572649572649573e-1,
a[22,2]=0.,
a[22,3]=0.,
a[22,4]=.1036484245439469320066334991708126036484245439469320066334991708126036484245439469320e-3,
a[22,5]=.1275672917463962240081643066717693583365225156269932389335374410001275672917463962240,
a[22,6]=0.,
a[22,7]=.1146176189444805429210887300555916540321998411667027651433109522778138762544220633889,
a[22,8]=0.,
a[22,9]=-.1316367023754075454122030740568234746157428970656730321378667908709827666511411271542,
a[22,10]=-.3845365626455519329296693060083837913367489520260829063809967396367023754075454122031,
a[22,11]=0.,
a[22,12]=0.,
a[22,13]=0.,
a[22,14]=0.,
a[22,15]=0.,
a[22,16]=0.,
a[22,17]=0.,
a[22,18]=0.,
a[22,19]=.3845365626455519329296693060083837913367489520260829063809967396367023754075454122031,
a[22,20]=.1316367023754075454122030740568234746157428970656730321378667908709827666511411271542,
a[22,21]=-.1146176189444805429210887300555916540321998411667027651433109522778138762544220633889,
a[23,1]=.4938271604938271604938271604938271604938271604938271604938271604938271604938271604938e-1,
a[23,2]=.3950617283950617283950617283950617283950617283950617283950617283950617283951,
a[23,3]=0.,
a[23,4]=0.,
a[23,5]=0.,
a[23,6]=-.6985294117647058823529411764705882352941176470588235294117647058823529411764705882353,
a[23,7]=-.3834558992979908012587751149842653110626966836117162914548535463568143306705398208666,
a[23,8]=0.,
a[23,9]=0.,
a[23,10]=0.,
a[23,11]=0.,
a[23,12]=0.,
a[23,13]=0.,
a[23,14]=0.,
a[23,15]=0.,
a[23,16]=0.,
a[23,17]=0.,

a[23,18]=0.,
a[23,19]=0.,
a[23,20]=0.,
a[23,21]=.3834558992979908012587751149842653110626966836117162914548535463568143306705398208666,
a[23,22]=.6985294117647058823529411764705882352941176470588235294117647058823529411764705882353,
a[24,1]=.25,
a[24,2]=0.,
a[24,3]=-3179947624392068836513280957725402169846614291058735503179947624392068836513280957725,
a[24,4]=0.,
a[24,5]=0.,
a[24,6]=0.,
a[24,7]=0.,
a[24,8]=0.,
a[24,9]=0.,
a[24,10]=0.,
a[24,11]=0.,
a[24,12]=0.,
a[24,13]=0.,
a[24,14]=0.,
a[24,15]=0.,
a[24,16]=0.,
a[24,17]=0.,
a[24,18]=0.,
a[24,19]=0.,
a[24,20]=0.,
a[24,21]=0.,
a[24,22]=0.,
a[24,23]=.3179947624392068836513280957725402169846614291058735503179947624392068836513280957725,
a[25,1]=-8.372198884106881534693978918377517566370093845755858261918976097289568611513794096080,
a[25,2]=-3.465,
a[25,3]=-2.497530864197530864197530864197530864197530864197530864197530864197530864197530864198,
a[25,4]=0.,
a[25,5]=0.,
a[25,6]=-1.6625,
a[25,7]=-1.835440144855967078189300411522633744855967078189300411522633744855967078189300411523,
a[25,8]=0.,
a[25,9]=-73.24196602168055417777290626459588169783506828087515911825626050817981149401335285663,
a[25,10]=.8828637638234425252214582684159081671841568649549683123402374725184174104751169772484,
a[25,11]=-8814190544577027565498395466693637610438359285341071841488888129094808522593697679385,
a[25,12]=-4803279491030982366096634651792419931607860681361863738107180364716221880854580858817,
a[25,13]=32.87910868410832718408338785621195992650771605408789380060168750394871935374300100483,
a[25,14]=52.97402230205695709773274200747992549975863857695768897511204041642006328750715675109,
a[25,15]=-8.886835693767662867753935454859468248687886743734693869236791808373822253213426903999,
a[25,16]=2.509716434086569985363077956674238286724267256774032757480361264001771307725305753519,
a[25,17]=1.162475886937088360535666476065014131326443901611275096836079807311057386735832665899,
a[25,18]=.5839488303468939449260909132929399737902477241002934521537006256042283009612379572628,
a[25,19]=1.581,
a[25,20]=1.33584,
a[25,21]=1.835440144855967078189300411522633744855967078189300411522633744855967078189300411523,
a[25,22]=1.6625,
a[25,23]=2.497530864197530864197530864197530864197530864197530864197530864197530864197530864198,
a[25,24]=3.465,

b[1]=.2380952380952380952380952380952380952380952380952380952380952380952380952380952380952380952380952e-1,
b[2]=-11,
b[3]=-17,
b[4]=0.,
b[5]=0.,

