

A modification Terry Feagin's 25 stage, order 12 Runge-Kutta scheme

The 25 stage Runge-Kutta scheme considered here is modification a scheme of Terry Feagin, University of Houston – Clear Lake, Houston, Texas, USA.

The scheme has the following values for specific nodes and weights.

$$c_2 = \frac{48}{107}, c_3 = \frac{883 - \sqrt{7849}}{1458}, c_4 = \frac{883 - \sqrt{7849}}{972}, c_5 = \frac{397 - \sqrt{7849}}{972}, c_6 = \frac{80}{81}, c_9 = \frac{9}{16}, c_{10} = \frac{5}{6}$$

$$c_{19} = \frac{5}{6}, c_{22} = \frac{397 - \sqrt{7849}}{972}, c_{23} = \frac{883 - \sqrt{7849}}{1458}, c_{24} = \frac{48}{107}, c_{25} = 1.$$

$$b_2 = \frac{3}{128}, b_3 = \frac{1}{32}, b_4 = 0, b_5 = \frac{1}{24}, b_6 = 0, b_7 = \frac{1}{20}, b_8 = \frac{1}{20}, b_9 = 0, b_{10} = \frac{1}{10}, b_{11} = \frac{1}{14},$$

$$b_{12} = 0, b_{18} = -\frac{1}{14}, b_{19} = -\frac{1}{10}, b_{20} = -\frac{1}{20}, b_{21} = -\frac{1}{20}, b_{22} = -\frac{1}{24}, b_{23} = -\frac{1}{32}, b_{24} = -\frac{3}{128}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx} P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

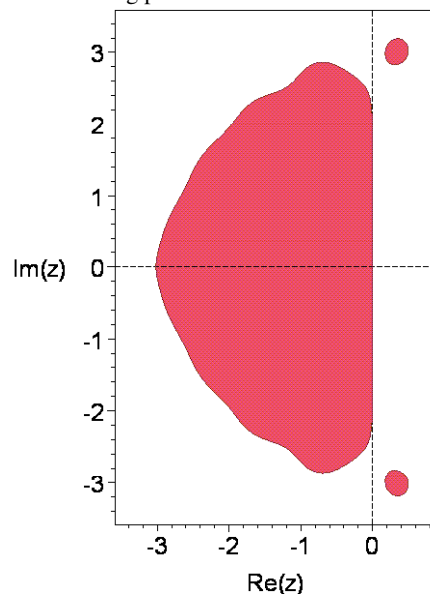
$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

$$\text{Also } c_8 = \frac{1}{8640} \left(r - \frac{2606183}{r} + 1691 \right), \text{ where } r = (3400884035 + 864\sqrt{39206776574697})^{\left(\frac{1}{3}\right)}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.1234250265 \times 10^{-6}$.

The maximum magnitude of the linking coefficients is: 9.954703775.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.0173, 0]$.

a[6,3]=0.,
a[6,4]=.4071694309883169298205672217378369342944057815568816310406296126697263808556072005379,
a[6,5]=.4883716804754181877657247975938623691547306093933179062708527465644479243568204048312,
a[7,1]=.9534922756660104865459638866208124038629642835587588783896764840713502540174683798834e-1,
a[7,2]=0.,
a[7,3]=0.,
a[7,4]=.1355746469520854542316809185518692829786912952549093201503141697149129528923677527486,
a[7,5]=.4730296531562839029210810917398318256058813722630299705804805524146746003867251353617,
a[7,6]=-.429760437616147372909767776453543743862677746651756778988841614139972085756675256926e-1,
a[8,1]=.1197621744744040418423084818651271945809975791630152562626874430872592612181493204123,
a[8,2]=0.,
a[8,3]=0.,
a[8,4]=0.,
a[8,5]=.2124767931560042175888371427239614483481358452930636033348233215664906780967911148238,
a[8,6]=.1212340257223529811315137341610073421123984651425834134692968953744849214272416520680e-1,
a[8,7]=-.5813876739937190933558651792218255021993521044821585173540793646391511143795799299884e-1,
a[9,1]=.9062385318129760410170832317173219032242410951281796663541576558337811618749213595740e-1,
a[9,2]=0.,
a[9,3]=0.,
a[9,4]=0.,
a[9,5]=0.,
a[9,6]=-.5459512551141770660378613394067568445427061958274102952880326173186195092976918494443e-2,
a[9,7]=.7196857817834543542239558709251532941841254196981766177204719277312608242209383300678e-1,
a[9,8]=.4053670811914987311362747031298200487045904104756384745454173678166819964833909495303,
a[10,1]=.8779068921444648762217654955879284722142196335696194975809735233779980381916443849904e-1,
a[10,2]=0.,
a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=.7095653374596866278579611963778936412879966934149647277713812725560398058291684969198e-2,
a[10,7]=.4368655538861312786196833374161271176727268109642122941262804438173137218026774399122,
a[10,8]=.4277733622610464100283100626723204064074889354258703911683342247476306412033704748473,
a[10,9]=-.1261919254028877092154162282776859743811843433478609489970925002949712315501707048945,
a[11,1]=.6625040299373659048512849852131922333337133767296861973370808560825661325824897896479e-1,
a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
a[11,5]=0.,
a[11,6]=.157929494870039986051087216372665848283329185139896707608218809553617687966985220233,
a[11,7]=1.759359496931358842573487043218007067018179983600472454945326036581939020547729934116,
a[11,8]=.5926204378018587363734684992506650590746731712661721929384235503751290220690961930849,
a[11,9]=-1.172743321756064244761441824076207503016389064033347604385630017013281836641066514068,
a[11,10]=-.4557235421166306695464362850971922246220302375809935205183585313174946004319654427646,
a[12,1]=.4437523738995657748838629239171545199639504920922948333134685659734533823582904232479e-1,
a[12,2]=0.,
a[12,3]=0.,
a[12,4]=0.,
a[12,5]=0.,
a[12,6]=-.4347892247733781595677993934677293794096240537632088760308083428212138244249612599121e-2,
a[12,7]=.1462777557716291332237909253984122865899408514833634019786961951626922849933177550450e-1,
a[12,8]=.2131346726458540881568602878489440087698897560731084195796891044403767446396095269067e-1,
a[12,9]=-.2672433161637370946274003076782422610616791118418465277057298042032529387835420256014e-1,
a[12,10]=.1889593737346470508840599271156701309218518018625995410986637872857335672830341476583e-2,
a[12,11]=.3677436301549776727081691620698712897294457578145521407932755450485946939847649067507e-2,
a[13,1]=.1945888036917512547331581309925970096305250890558108163133556402474238495411785837128e-1,
a[13,2]=0.,
a[13,3]=0.,
a[13,4]=0.,

a[13,5]=0.,
a[13,6]=0.,
a[13,7]=0.,
a[13,8]=0.,
a[13,9]=.6785154300702292993513742850429620775837422016092550399574130039824507784534502224276e-4,
a[13,10]=-.4298039382898988187744420353617260919945467265249210788061198433844184623624436442913e-4,
a[13,11]=.1763653382092351804920378970917718658696944900238270316625622748176808314883224121237e-4,
a[13,12]=.6538666380854245302456118290233042845821641966191005647397644441527378364477897109862e-1,
a[14,1]=.2068367011605371440266648193554846842052501540164651298853935084807220611313879523181,
a[14,2]=0.,
a[14,3]=0.,
a[14,4]=0.,
a[14,5]=0.,
a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,
a[14,9]=.1667965689202105042636587707802488213405138117676669699733355864054259085808713452203e-1,
a[14,10]=-.8795175320876273925326427705967576562888968045855379443047321715229391582619623973710e-2,
a[14,11]=.3466879850522708393855748656990969490018600083182743043716464888120260480163696653603e-2,
a[14,12]=-.8612639102098855084500248882467126016293639927509245072096981319667674760277089713036,
a[14,13]=.9086514508923237726265789299077964776601340896844429381749671919911341900821259573852,
a[15,1]=.1214692019624634818990932882840628925848671091630716607766715377904504107219479014111e-1,
a[15,2]=0.,
a[15,3]=0.,
a[15,4]=0.,
a[15,5]=0.,
a[15,6]=0.,
a[15,7]=0.,
a[15,8]=0.,
a[15,9]=.8587574891915415254996841655064416938684257946369205707727904699083604926056882463372e-1,
a[15,10]=-.1863518796254172704037315128177618110722947071918741336510275915861249085624311911381e-1,
a[15,11]=.6350208100584190684210494332600713240879900544889406182012497311204426233420888786723e-2,
a[15,12]=.1356604798468768842143536326912214112551988156537784695961988202948385226095563011860,
a[15,13]=-.2645778419345038750783399462121551923188368999578634291766829953267775747096412242508e-1,
a[15,14]=.3050596150931305389097652735001191171977051541363066573496135403153662091514664367914,
a[16,1]=.2281063744721816899671060261223898551508176248601898833204870656180023092448522329191,
a[16,2]=0.,
a[16,3]=0.,
a[16,4]=0.,
a[16,5]=0.,
a[16,6]=0.,
a[16,7]=0.,
a[16,8]=0.,
a[16,9]=-.4973821989528795811518324607329842931937172774869109947643979057591623036649214659686,
a[16,10]=.1346749226006191950464396284829721362229102167182662538699690402476780185758513931889,
a[16,11]=-.3869301866457546666833482550157589090948074842016329912971019172379293408996771686808e-1,
a[16,12]=-.1272726616182261517823886285256361793466245002129771797444255027962559722973505406401,
a[16,13]=1.437067715300586975575039572132012239676512916392870702116996029431050437948391518733,
a[16,14]=-.213238808243665824778627404962143265263114331649232495685554701809429635640584046896,
a[16,15]=.9566160264053516367359816906700741764851153375506741262678011900112049235819217034842,
a[17,1]=1.706515982887603449108125193256345362390496762740107141227298057019894197484300794976,
a[17,2]=0.,
a[17,3]=0.,
a[17,4]=0.,
a[17,5]=0.,
a[17,6]=0.,
a[17,7]=0.,
a[17,8]=0.,

a[17,9]=2.139059820794698423487620833522356133834648813995746687313450159782754836445196714930,
a[17,10]=.6289682905532989143877921043705734947087608480770236332444711704282583742651458563390,
a[17,11]=- .4887787307610143643345724347210014570022107335849698834865297116271814777012455077458e-1,
a[17,12]=-7.474965144232589698190965594199344040590348167595508441152320929688039217304770144991,
a[17,13]=6.528737266898819180907874989939628211667666928401297052185530968474373510552258837780,
a[17,14]=- .3991665531786491975790352665705619831043711050984402292823890772666833355009619481277,
a[17,15]=-1.234216852823573674175620909111326754472186370771809685940424294147193093428297678703,
a[17,16]=- .9309429896842224965763180007518377089408614539539211234515564774242048646564026437970,
a[18,1]=2.616700085606639672439046195045140588883776712333482322003880341923024661730442503778,
a[18,2]=0.,
a[18,3]=0.,
a[18,4]=0.,
a[18,5]=0.,
a[18,6]=.1579294948700399860510872163726658482833329185139896707608218809553617687966985220233,
a[18,7]=1.759359496931358842573487043218007067018179983600472454945326036581939020547729934116,
a[18,8]=.5926204378018587363734684992506650590746731712661721929384235503751290220690961930849,
a[18,9]=5.348229362219277914827989554700720529167056378044914381692935880104620934969185781697,
a[18,10]=.1900208364044018456203874623472949920523253568879151195030029536412916851164959876199,
a[18,11]=.7225278538701025398389619984217430067329350575879897314184536350103406873614604659789e-1,
a[18,12]=-9.954703774576620123157243033016527526097253233678212104687288441823386733582882493514,
a[18,13]=7.713650104801853735850610052900032467556928790378767414191912225551316274791985373218,
a[18,14]=.4273899818304534283248900005343175033313647755967464674657951300287133073650537256677,
a[18,15]=-5.901557691014728282451879109332224767064773259021467831877789215857104535922010100722,
a[18,16]=-1.945847871077682609386877500442979828951276304739927969839803849315281125255129667880,
a[18,17]=- .1283502804595641598735694332300287638564906855023892767647708504767483617640701343299,
a[19,1]=1.266749517822939706798362066705113373367104251099996972341526842847454802802353857428,
a[19,2]=0.,
a[19,3]=0.,
a[19,4]=0.,
a[19,5]=0.,
a[19,6]=.7095653374596866278579611963778936412879966934149647277713812725560398058291684969198e-2,
a[19,7]=.4368655538861312786196833374161271176727268109642122941262804438173137218026774399122,
a[19,8]=.4277733622610464100283100626723204064074889354258703911683342247476306412033704748473,
a[19,9]=-1.322536499885687881456688984918078121376173742071730806839284240656600063089233166347,
a[19,10]=.6641972515926384837195895717318623416004376311423479961643163259395784275136673857005,
a[19,11]=- .1447653155812597995984332973159126397082914209874549946235396852079965059183373865664,
a[19,12]=-6.431778531615778807054999818601295693466152640303707424934258682220563785128087220807,
a[19,13]=6.188608927981319042177632416847901050136517670720162294584781401749269596604399982681,
a[19,14]=-1.599539437185161074824705276166190051803637618810115377036219334754209793979085228917,
a[19,15]=2.023530305654945612467820166366457350495763057839686297128768519246597543562528470217,
a[19,16]=- .6751056985996472474703476539979310019402192318743406794345320907560858535829195910767,
a[19,17]=- .1155282027891295549883315528053823872729803823479458048382876243272480066172250012480e-2,
a[19,18]=- .66064743444857960801585553842765910592380532922263818542171327901343316450121118696322e-2,
a[20,1]=.9262636204394898113786984120492695274406538411843612068733928916393866040396774385819,
a[20,2]=0.,
a[20,3]=0.,
a[20,4]=0.,
a[20,5]=.2124767931560042175888371427239614483481358452930636033348233215664906780967911148238,
a[20,6]=.1212340257223529811315137341610073421123984651425834134692968953744849214272416520680e-1,
a[20,7]=- .5813876739937190933558651792218255021993521044821585173540793646391511143795799299884e-1,
a[20,8]=0.,
a[20,9]=.1489244841744261844647174034424681041928182905273442424533869148181301680608275392282,
a[20,10]=- .4978792048465214977626636424197530625078857536429469648258747092185449685579197151269,
a[20,11]=- .4791343047827829461765435469081753396423632636280868931479768426932485112364453208481e-1,
a[20,12]=-3.590246881655962120727772985084191507347179823574020896443603845934939047687310762820,
a[20,13]=3.082569353120583482943465132264300377671746027354226399822309044748233460342232807703,
a[20,14]=- .3609918750459103621545569054852264264327025901900778351125974295249912171504676983574,
a[20,15]=- .5072941703350940183109038240108493755350491717336156136060222706098380306525953149769e-1,

a[20,16]=- .8165576232082414735862424577977005236295500019710421783267446996669208782982559084778e-3,
a[20,17]=.2754506719562100258133965001438778344780411776638489708063272452539193545568291754888e-5,
a[20,18]=.6994625090618900018151927334048129365522609368986984005927913922377860074995116677942e-1,
a[20,19]=.4406330780103859193409746393424776211713475069406239178004140300937286972948152956231,
a[21,1]=.9534922756660104865459638866208124038629642835587588783896764840713502540174683798834e-1,
a[21,2]=0.,
a[21,3]=0.,
a[21,4]=.1355746469520854542316809185518692829786912952549093201503141697149129528923677527486,
a[21,5]=.4730296531562839029210810917398318256058813722630299705804805524146746003867251353617,
a[21,6]=- .429760437616147372909767776453543743862677746651756778988841614139972085756675256926e-1,
a[21,7]=0.,
a[21,8]=- .4522484351634166780066038434594443966478247302042176248894535756148235872864827792387,
a[21,9]=0.,
a[21,10]=- .2201241105092869311411191148702857874367386414129922607262399039463376333837905375180,
a[21,11]=- .3393023183947587624755272930275465806558306466549241634246387464428084998492264479733e-1,
a[21,12]=0.,
a[21,13]=0.,
a[21,14]=0.,
a[21,15]=0.,
a[21,16]=0.,
a[21,17]=0.,
a[21,18]=.3393023183947587624755272930275465806558306466549241634246387464428084998492264479733e-1,
a[21,19]=.2201241105092869311411191148702857874367386414129922607262399039463376333837905375180,
a[21,20]=.4522484351634166780066038434594443966478247302042176248894535756148235872864827792387,
a[22,1]=.1872264956149438210842500878346890072136977423754098485409302311756129403115242049409,
a[22,2]=0.,
a[22,3]=.2054210710613901257723003060745111718325548313971443548912763788003574945824128604360,
a[22,4]=- .7535804221865051010555140073598692551780681886520240821009316091593474558083709062486e-1,
a[22,5]=0.,
a[22,6]=0.,
a[22,7]=.1017020051661452922602761373610974718538039007485642334530319241601431682941003146638,
a[22,8]=- .1782588207072600304947031980443547513103490410530912115246574462743302012311218491261,
a[22,9]=0.,
a[22,10]=.6351671858662875155872049194183746696022763821535117787260665881271980570357764444783,
a[22,11]=.1006680908505391743860533396865017354064606181017411605574645809327497353192078744684,
a[22,12]=0.,
a[22,13]=0.,
a[22,14]=0.,
a[22,15]=0.,
a[22,16]=0.,
a[22,17]=0.,
a[22,18]=- .1006680908505391743860533396865017354064606181017411605574645809327497353192078744684,
a[22,19]=- .6351671858662875155872049194183746696022763821535117787260665881271980570357764444783,
a[22,20]=.1782588207072600304947031980443547513103490410530912115246574462743302012311218491261,
a[22,21]=- .1017020051661452922602761373610974718538039007485642334530319241601431682941003146638,
a[23,1]=.2139710170164903579187027570797016260028637717668568628582423582660728993470884095552,
a[23,2]=.3308886659552985999152965717024405430161000648380443339564999411072842268616449069462,
a[23,3]=0.,
a[23,4]=0.,
a[23,5]=- .6827104755423165632490010068267867464715542450926482047778865509399643106869000252479,
a[23,6]=0.,
a[23,7]=- .4054274516554988728231222831042925648239619737543049929538872781740392835494215421390,
a[23,8]=0.,
a[23,9]=0.,
a[23,10]=0.,
a[23,11]=0.,
a[23,12]=0.,
a[23,13]=0.,

a[23,14]=0.,
a[23,15]=0.,
a[23,16]=0.,
a[23,17]=0.,
a[23,18]=0.,
a[23,19]=0.,
a[23,20]=0.,
a[23,21]=.4054274516554988728231222831042925648239619737543049929538872781740392835494215421390,
a[23,22]=.6827104755423165632490010068267867464715542450926482047778865509399643106869000252479,
a[24,1]=.4485981308411214953271028037383177570093457943925233644859813084112149532710280373832,
a[24,2]=0.,
a[24,3]=-.2504558166750941086485041189158855136665236744746146886041710992135690482491410320400,
a[24,4]=0.,
a[24,5]=0.,
a[24,6]=0.,
a[24,7]=0.,
a[24,8]=0.,
a[24,9]=0.,
a[24,10]=0.,
a[24,11]=0.,
a[24,12]=0.,
a[24,13]=0.,
a[24,14]=0.,
a[24,15]=0.,
a[24,16]=0.,
a[24,17]=0.,
a[24,18]=0.,
a[24,19]=0.,
a[24,20]=0.,
a[24,21]=0.,
a[24,22]=0.,
a[24,23]=.2504558166750941086485041189158855136665236744746146886041710992135690482491410320400,
a[25,1]=1.195075973788434669193745228091851405012361920183222411956772556636023858154099627912,
a[25,2]=.5427862149532710280373831775700934579439252336448598130841121495327102803738317757009,
a[25,3]=.3508292215599812296420046389156136006471257223951183450859198062841117399807893186775,
a[25,4]=0.,
a[25,5]=.2986858330497634964214379404867192015813049822280335895903253660362343859255187610460,
a[25,6]=0.,
a[25,7]=.3578022625248650869907338390598378377850439671021371226530935104466459834271865321925,
a[25,8]=.2372557840322494615821093239832316157135373176317785342250286730014976425025390519954,
a[25,9]=5.894516309580471947145035328358842656339263918357828838206343968082542051759718267588,
a[25,10]=.4239524793746901662652117251353525137270027572260516378646432677384573053875365282407,
a[25,11]=.3083611483235031041585219306593180438569623531052209934221341769330077857097458781721,
a[25,12]=-3.383660511628749558466035954376668278602730141784687348149186361211633184162773253816,
a[25,13]=1.975657560035959653255402278286619645002895919845554139903769438620111511619345671942,
a[25,14]=1.594838040963513792876773150068439683273446224226243744151818177962312598767999453128,
a[25,15]=-5.690752303854665628891694282060956426336510730055461638304475551183867305177452475959,
a[25,16]=-0.8549258456400315328738710983353547523110591527985262501617649452122740708202002748753,
a[25,17]=.1035864597296165801174454397803349763875604480081139989607377884990759107091795663713,
a[25,18]=-0.1426968312980530265153220204724269526221907590875088899861492491252991565596624604639,
a[25,19]=-0.4239524793746901662652117251353525137270027572260516378646432677384573053875365282407,
a[25,20]=-0.2372557840322494615821093239832316157135373176317785342250286730014976425025390519954,
a[25,21]=-0.3578022625248650869907338390598378377850439671021371226530935104466459834271865321925,
a[25,22]=-0.2986858330497634964214379404867192015813049822280335895903253660362343859255187610460,
a[25,23]=-0.3508292215599812296420046389156136006471257223951183450859198062841117399807893186775,
a[25,24]=-0.5427862149532710280373831775700934579439252336448598130841121495327102803738317757009,

