

A modification Terry Feagin's 25 stage, order 12 Runge-Kutta scheme

The 25 stage Runge-Kutta scheme considered here is modification a scheme of Terry Feagin, University of Houston – Clear Lake, Houston, Texas, USA.

The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{35}{76}, c_3 = \frac{5}{9}, c_4 = \frac{5}{6}, c_5 = \frac{1}{3}, c_6 = 1, c_9 = \frac{9}{16}, c_{10} = \frac{5}{6}, c_{19} = \frac{5}{6}, c_{22} = \frac{1}{3}, c_{23} = \frac{5}{9}, c_{24} = \frac{35}{76}, c_{25} = 1,$$

$$b_2 = \frac{3}{128}, b_3 = \frac{1}{32}, b_4 = 0, b_5 = \frac{1}{24}, b_6 = 0, b_7 = \frac{1}{20}, b_8 = \frac{1}{20}, b_9 = 0, b_{10} = \frac{1}{10}, b_{11} = \frac{1}{14},$$

$$b_{12} = 0, b_{18} = -\frac{1}{14}, b_{19} = -\frac{1}{10}, b_{20} = -\frac{1}{20}, b_{21} = -\frac{1}{20}, b_{22} = -\frac{1}{24}, b_{23} = -\frac{1}{32}, b_{24} = -\frac{3}{128}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative $P'_6(x) = \frac{d}{dx} P_6(x)$ of the **Legendre polynomial** $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$.

They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

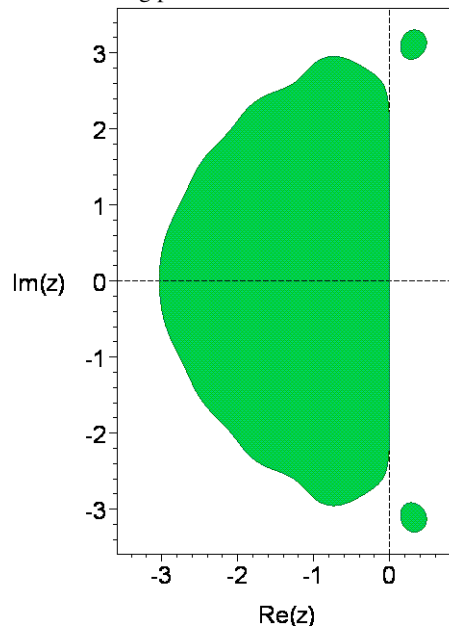
In addition

$$c_8 = \frac{(3400884035 + 864\sqrt{39206776574697}) \left(\frac{1}{3}\right)}{8640} - \frac{2606183}{8640(3400884035 + 864\sqrt{39206776574697}) \left(\frac{1}{3}\right)} + \frac{1691}{8640}.$$

The principal error norm, that is, the 2-norm of the principal error terms is: $0.1125820716 \times 10^{-6}$.

The maximum magnitude of the linking coefficients is: 12.92133273.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.0305, 0]$.

a[6,3]=0.,
a[6,4]=.4,
a[6,5]=.5,
a[7,1]=.1033644716500104775703954356904817915433427083303498792441969232007020836692076572831,
a[7,2]=0.,
a[7,3]=0.,
a[7,4]=.1240530945289467610615818892371153282110747849551802980440742772315189064222869368349,
a[7,5]=.4831711675610328992888364804519625087241092575172891773023800341818228904190296838151,
a[7,6]=-.3875302456947632520856814437676205803957333023413680388042903644485548362506013679914e-1,
a[8,1]=.1240382614318333240819045859801751681400246706986336122924798440448279924947470898167,
a[8,2]=0.,
a[8,3]=0.,
a[8,4]=0.,
a[8,5]=.2170506321979584863178462569531599428759163537577341676846566744497954872579456743665,
a[8,6]=.1374557920759667598129078018350481905944439909394085308429177535450188051849310154998e-1,
a[8,7]=-.6610953172676828444558313414981495316726682520850165659175461032462670124939971157633e-1,
a[9,1]=.9147748948568829831449918469804321970888320999766601000904864070801948953424558529918e-1,
a[9,2]=0.,
a[9,3]=0.,
a[9,4]=0.,
a[9,5]=0.,
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a[9,8]=.4083943155826410467273068526538947800933031856649246445512393508534091461041598611744,
a[10,1]=.8900136525025510189545093554238417801432326974034341186926989993223983196257058017745e-1,
a[10,2]=0.,
a[10,3]=0.,
a[10,4]=0.,
a[10,5]=0.,
a[10,6]=.4995282266455323601977934084206928004058911494068140919558100299611985086558381414011e-2,
a[10,7]=.3979182388198289973417396030013471560834350609314249708263044040250700525391549655752,
a[10,8]=.4279302107525766110681926083008979815582407305803964063123594678423525630853815189010,
a[10,9]=-.8651176375578270057402774759550291032672463941289959659415853876594109934033211273435e-1,
a[11,1]=.690377407065398842620189840898622406089037326887686735299824947744072559861130193951e-1,
a[11,2]=0.,
a[11,3]=0.,
a[11,4]=0.,
a[11,5]=0.,
a[11,6]=.1316633083077899368614323951690414312668647200728382264527827130788686729110294170782,
a[11,7]=.1559452296070784420981759080324071365866106166789649899338117387179237025915899302256,
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a[12,3]=0.,
a[12,4]=0.,
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a[12,9]=-.2504548470127629763146462537418449854581595401019870907778688235386017465098008705307e-1,
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a[13,4]=0.,

a[13,5]=0.,
a[13,6]=0.,
a[13,7]=0.,
a[13,8]=0.,
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a[14,2]=0.,
a[14,3]=0.,
a[14,4]=0.,
a[14,5]=0.,
a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,
a[14,9]=.1667960671041564728280458666646964503063265050947925052155140101473800468812591484420e-1,
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a[15,4]=0.,
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a[15,6]=0.,
a[15,7]=0.,
a[15,8]=0.,
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a[16,3]=0.,
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a[16,5]=0.,
a[16,6]=0.,
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a[16,14]=-.2186363857316288833697544316228840270262274175740904891538175305844302801928609887278,
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a[17,2]=0.,
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a[17,4]=0.,
a[17,5]=0.,
a[17,6]=0.,
a[17,7]=0.,
a[17,8]=0.,

a[17,9]=2.056868999755680098931381768280965089314432490847256988111636852647176490407241749871,
a[17,10]=.6215970772448139207351432357900441632191446367788775697801237591517890531028042308812,
a[17,11]=-.4513176102786270645357954850055907997597231196209747090142051792519402270622334083263e-1,
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a[17,14]=-.6362174826642236803028930139185802971311098466714699884669577414976276618668688400830,
a[17,15]=-1.082770086286099664788184252368387950177314779571295386488813114564680253443675444849,
a[17,16]=-.9295290375792039997783972382912332142207880575118997475070736339961169400050824335694,
a[18,1]=3.255961880979495029108776251737021471207243125872841579605541601947739281585786837640,
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a[18,3]=0.,
a[18,4]=0.,
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a[18,14]=.1280638204574762585055630115843668028891465764738484697414825567186225812466452312522e-1,
a[18,15]=-5.608839309906026530453185844628969610944010472899781426928970420986750897852725432876,
a[18,16]=-1.943009352428196108838337767823642877287248991241669204778734219860256733548749791369,
a[18,17]=-1.283526018494045420184287143193446207421464913356123535599232759641139856465371475959,
a[19,1]=1.440511404868138100571841509581624834119951843718332510524214241379130017299733473912,
a[19,2]=0.,
a[19,3]=0.,
a[19,4]=0.,
a[19,5]=0.,
a[19,6]=.4995282266455323601977934084206928004058911494068140919558100299611985086558381414011e-2,
a[19,7]=.3979182388198289973417396030013471560834350609314249708263044040250700525391549655752,
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a[19,10]=.6588353074054981585680120069850677901803719118855862845919334052018439603236955395540,
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a[19,12]=-7.224651155157847941792969762120260510931173391318485047058880156264912366661918313496,
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a[19,14]=-1.700832864140321560638929113272101872631766326268970098272485293337998314407805271317,
a[19,15]=2.074294075723052913906091486669800417347535942724345721880263601813825689508291009328,
a[19,16]=-0.6747439626443064718629581295708377231920798759984050586488916899925211549437997031744,
a[19,17]=-1.156188347949395004907036084359076100596057549353055820457292017095159857577078205324e-2,
a[19,18]=-0.7276507276507276507276507276507276507276507276507276507276507276507276507276507e-2,
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a[20,3]=0.,
a[20,4]=0.,
a[20,5]=.2170506321979584863178462569531599428759163537577341676846566744497954872579456743665,
a[20,6]=.1374557920759667598129078018350481905944439909394085308429177535450188051849310154998e-1,
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a[20,9]=.1521880339447512760838219396331281613277831448746071614605224726345410644794515149620,
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a[20,13]=3.159823101897254334762383114951690002143573651233127409778343340492145895532387373153,
a[20,14]=-0.3702065236411917685634106268451378901067530327436914675647768285998744402601246957522,
a[20,15]=-0.5147348602758037542430441565973184194317084177157368187824364981252678394750889429481e-1,

