

## Terry Feagin's 25 stage, order 12 Runge-Kutta scheme

The 25 stage Runge-Kutta scheme considered here is that of Terry Feagin, University of Houston – Clear Lake, Houston, Texas, USA  
The coefficients (correct to 60 digits) are available at: <http://sce.uhcl.edu/rungekutta>

The scheme has the following rational values for specific nodes and weights.

$$c_2 = \frac{1}{5}, c_3 = \frac{5}{9}, c_4 = \frac{5}{6}, c_5 = \frac{1}{3}, c_6 = 1, c_9 = \frac{9}{16}, c_{10} = \frac{5}{6}, c_{19} = \frac{5}{6}, c_{22} = \frac{1}{3}, c_{23} = \frac{5}{9}, c_{24} = \frac{1}{5}, c_{25} = 1,$$

$$b_2 = \frac{3}{128}, b_3 = \frac{1}{32}, b_4 = 0, b_5 = \frac{1}{24}, b_6 = 0, b_7 = \frac{1}{20}, b_8 = \frac{1}{20}, b_9 = 0, b_{10} = \frac{1}{10}, b_{11} = \frac{1}{14},$$

$$b_{12} = 0, b_{18} = -\frac{1}{14}, b_{19} = -\frac{1}{10}, b_{20} = -\frac{1}{20}, b_{21} = -\frac{1}{20}, b_{22} = -\frac{1}{24}, b_{23} = -\frac{1}{32}, b_{24} = -\frac{3}{128}.$$

The nodes:

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66}, c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{15} = \frac{1}{2}, c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66}, c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}$$

are the zeros of the derivative  $P'_6(x) = \frac{d}{dx}P_6(x)$  of the **Legendre polynomial**  $P_6(x)$  mapped linearly from the interval  $[-1, 1]$  to the interval  $[0, 1]$ .

They provide nodes for **Gauss-Lobatto integration** on the interval  $[0, 1]$ .

These values give rise to the following weights:

$$b_1 = \frac{1}{42}, b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{15} = \frac{128}{525}, b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100}, b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100}, b_{25} = \frac{1}{42}.$$

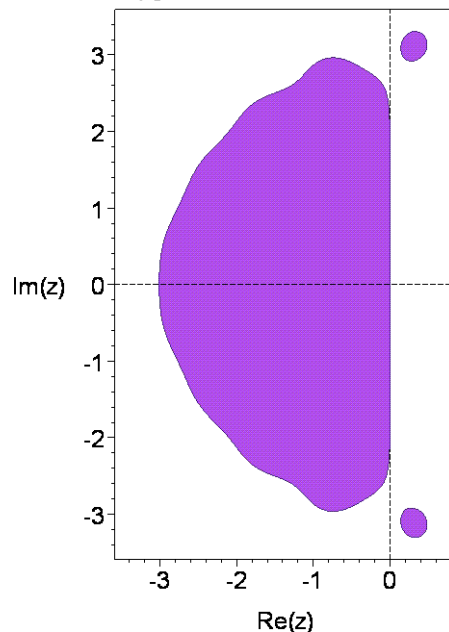
In addition

$$c_8 = \frac{(3400884035 + 864\sqrt{39206776574697})\left(\frac{1}{3}\right) - 2606183}{8640} - \frac{2606183}{8640(3400884035 + 864\sqrt{39206776574697})\left(\frac{1}{3}\right)} + \frac{1691}{8640}.$$

The principal error norm, that is, the 2-norm of the principal error terms is:  $0.1367113081 \times 10^{-6}$ .

The maximum magnitude of the linking coefficients is: 12.37299734.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is  $[-3.0113, 0]$ .



a[6,4]=.4,  
a[6,5]=.5,  
a[7,1]=.1033644716500104775703954356904817915433427083303498792441969232007020836692076572831,  
a[7,2]=0.,  
a[7,3]=0.,  
a[7,4]=.1240530945289467610615818892371153282110747849551802980440742772315189064222869368349,  
a[7,5]=.4831711675610328992888364804519625087241092575172891773023800341818228904190296838151,  
a[7,6]=-.3875302456947632520856814437676205803957333023413680388042903644485548362506013679914e-1,  
a[8,1]=.1240382614318333240819045859801751681400246706986336122924798440448279924947470898167,  
a[8,2]=0.,  
a[8,3]=0.,  
a[8,4]=0.,  
a[8,5]=.2170506321979584863178462569531599428759163537577341676846566744497954872579456743665,  
a[8,6]=.1374557920759667598129078018350481905944439909394085308429177535450188051849310154998e-1,  
a[8,7]=-.6610953172676828444558313414981495316726682520850165659175461032462670124939971157633e-1,  
a[9,1]=.9147748948568829831449918469804321970888320999766601000904864070801948953424558529918e-1,  
a[9,2]=0.,  
a[9,3]=0.,  
a[9,4]=0.,  
a[9,5]=0.,  
a[9,6]=-.5443485237174696899657549441448386113461568738470091780683178340728768152294577966066e-2,  
a[9,7]=.6807168016884535185785151208951038631127517307587943722039518677930013251388913149245e-1,  
a[9,8]=.4083943155826410467273068526538947800933031856649246445512393508534091461041598611744,  
a[10,1]=.8900136525025510189545093554238417801432326974034341186926989993223983196257058017745e-1,  
a[10,2]=0.,  
a[10,3]=0.,  
a[10,4]=0.,  
a[10,5]=0.,  
a[10,6]=.4995282266455323601977934084206928004058911494068140919558100299611985086558381414011e-2,  
a[10,7]=.3979182388198289973417396030013471560834350609314249708263044040250700525391549655752,  
a[10,8]=.4279302107525766110681926083008979815582407305803964063123594678423525630853815189010,  
a[10,9]=-.8651176375578270057402774759550291032672463941289959659415853876594109934033211273435e-1,  
a[11,1]=.6950876241349075431126939064098098227060210616855446152557575792953755238456142607555e-1,  
a[11,2]=0.,  
a[11,3]=0.,  
a[11,4]=0.,  
a[11,5]=0.,  
a[11,6]=.1291469419001764619707595794827465511228717515014826340454871396367885195565200865413,  
a[11,7]=.1530736381023112950763425661432149390311775041124338743130106194678206265388420534343,  
a[11,8]=.5778747611291400525467513494545767153348921004185718827180365180125884787253599132116,  
a[11,9]=-.9512947723210889805323408373888594539309244987992286480509495701319828569591060982197,  
a[11,10]=-.408276642965631951497484981519757463459627174520978426909934,  
a[12,1]=.44486140329513586626945350709246358162016550101868415293331301157328390678444631360662e-1,  
a[12,2]=0.,  
a[12,3]=0.,  
a[12,4]=0.,  
a[12,5]=0.,  
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a[12,7]=.1069550640296242007212626028090591544692060776449573995939720279979491092361490435771e-1,  
a[12,8]=.2096162444999043332966742059289199208067346506600398980746522493272461502026769432938e-1,  
a[12,9]=-.2331460232593217866485614315519780776653378187560536038988470502127295139273620447039e-1,  
a[12,10]=.263265981064536974369934736325334761174975280887405725010964e-2,  
a[12,11]=.315472768977025060103545855572111407955208306374459723959783e-2,  
a[13,1]=.1945888151197554755888010965253177612420737620162731862312147414806601584184678965302e-1,  
a[13,2]=0.,  
a[13,3]=0.,  
a[13,4]=0.,  
a[13,5]=0.,

a[13,6]=0.,  
a[13,7]=0.,  
a[13,8]=0.,  
a[13,9]=.6785129491718125093061216534523674761943647812591653323215344531729041102383826068881e-4,  
a[13,10]=-.4297958590492736232710053302301623435688633877248836036755502109954489170536483550959e-4,  
a[13,11]=.1763589822602851554074859289533021399375534428299757341489807141089039042639321498044e-4,  
a[13,12]=.6538666274150270510095952313851810335495113587873820983519242333986308816206310607574e-1,  
a[14,1]=.2068368356642771059168281747982723610789091960434464115982313836045688158444651960319,  
a[14,2]=0.,  
a[14,3]=0.,  
a[14,4]=0.,  
a[14,5]=0.,  
a[14,6]=0.,  
a[14,7]=0.,  
a[14,8]=0.,  
a[14,9]=.1667960671041564728280458666646964503063265050947925052155140101473800468812591484420e-1,  
a[14,10]=-.8795015632007102144570241782499865911302349902199592087049794487571848499522935439738e-2,  
a[14,11]=.3466754553624639108244623152463792094275136540985964036372314865670539326325668110382e-2,  
a[14,12]=-.8612644601057176781614325622583512420302704989668912017992248333482802709963824294591,  
a[14,13]=.9086518820740502810962394784692621450349571299392567891787847986693969945784247315140,  
a[15,1]=.2039260846544840100915113146769256860385044495624130045623817918441487821545443195970e-1,  
a[15,2]=0.,  
a[15,3]=0.,  
a[15,4]=0.,  
a[15,5]=0.,  
a[15,6]=0.,  
a[15,7]=0.,  
a[15,8]=0.,  
a[15,9]=.8694693920166859486754005555839475058339544609309409595773466506310797056354130304882e-1,  
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a[15,12]=.9874761281274347809037985286740338997389249680066322014454609882039267630836494384605e-1,  
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a[15,14]=.3011678640109679168370913038170516769200592297849574799980774547109414597317226366896,  
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a[16,2]=0.,  
a[16,3]=0.,  
a[16,4]=0.,  
a[16,5]=0.,  
a[16,6]=0.,  
a[16,7]=0.,  
a[16,8]=0.,  
a[16,9]=-.498707400793025250635016567442511512138603770959682292383042,  
a[16,10]=.134841168335724478552596703792570104791700727205981058201689,  
a[16,11]=-.3874582440558341584399042269240292309351610591428068056743603625175611891589978761354e-1,  
a[16,12]=-.1.274732574734748442403884308249089523809792927132503501996417388213689856389637269712,  
a[16,13]=1.439163644628771652011844524370380818752993035779118396305246406270252915801577043206,  
a[16,14]=-.2140074679679902542195035408273495696390280923448127954990268736031526728285351786283,  
a[16,15]=.9582024177544302398927241391097813710599088746051536487680380575432052006086080693288,  
a[17,1]=2.002224776559742036142496460125067471214403062257117212098013934570904909077811687956,  
a[17,2]=0.,  
a[17,3]=0.,  
a[17,4]=0.,  
a[17,5]=0.,  
a[17,6]=0.,  
a[17,7]=0.,  
a[17,8]=0.,  
a[17,9]=2.067018099615249120919546564381385958254118596733416006795539073419461189789965414880,

a[17,10]=.6239781360861395419574712798314944661552923161670210806631400641515876319986835645504,  
a[17,11]=-.4622836855003114302832035541290620693919471018801127231857717578220394222019200167241e-1,  
a[17,12]=-8.849732883626496148600752467271189492866048354570927010946452349296049980461005208984,  
a[17,13]=7.742577078508559762274372257918355895601885907850371974336285358245514323924977674267,  
a[17,14]=-5.883585192508692109933533141277117456441258821309412028964590026164853179425282383760,  
a[17,15]=-1.106837333623806493957047080169530561761957696170148994429009662680169614076285221420,  
a[17,16]=.9295290375792039997783972382912332142207880575118997475070736339961169400050824335694,  
a[18,1]=3.137895334120734429344516089898887968081612593303221002683164397131264854080655887290,  
a[18,2]=0.,  
a[18,3]=0.,  
a[18,4]=0.,  
a[18,5]=0.,  
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a[18,7]=1.530736381023112950763425661432149390311775041124338743130106194678206265388420534343,  
a[18,8]=.5778747611291400525467513494545767153348921004185718827180365180125884787253599132116,  
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a[18,11]=.7592929955789135601623013117852518735618013423331948952920597912045137042432676518814e-1,  
a[18,12]=-12.37299733801865132874145534025958065913498226175359059762562192965479091087793342733,  
a[18,13]=9.854558834647695439359572093173692020803677657217771019069796164343105374428968973275,  
a[18,14]=.8591114313704365295793577090523677728899804951223296011591248114960893346142179981950e-1,  
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a[18,16]=-1.943009352428196108838337767823642877287248991241669204778734219860256733548749791369,  
a[18,17]=-1.283526018494045420184287143193446207421464913356123535599232759641139856465371475959,  
a[19,1]=1.383600544321960148785381182981677168251632684899225199955650618232490683056428985354,  
a[19,2]=0.,  
a[19,3]=0.,  
a[19,4]=0.,  
a[19,5]=0.,  
a[19,6]=.4995282266455323601977934084206928004058911494068140919558100299611985086558381414011e-2,  
a[19,7]=.3979182388198289973417396030013471560834350609314249708263044040250700525391549655752,  
a[19,8]=.4279302107525766110681926083008979815582407305803964063123594678423525630853815189010,  
a[19,9]=-1.302991074244757709165514391230475733420714759983996459821461482811069921735213561975,  
a[19,10]=.6612922786693770290971125281075130727345734122940080715006994611067693544004895081155,  
a[19,11]=-1.445597743069543497659693936887034639005858224415456555301446467509926053698427625331,  
a[19,12]=-6.965760347317982034678538674610839193567922481059192554608250635478358752496696344633,  
a[19,13]=6.658085432359917483534082955422104506321931975769351207164429345917517820065930603070,  
a[19,14]=-1.669973751088414864046958057255108450498079691992362275757972756183370386412619176110,  
a[19,15]=2.064137023180352638322890403018326471306046512239864521700896369142928855915137997534,  
a[19,16]=-6.747439626443064718629581295708377231920798759984050586488916899925211549437997031744,  
a[19,17]=-1.156188347949395004907036084359076100596057549353055820457292017095159857577078205324e-2,  
a[19,18]=-5.44057908677007389319819914241631024660726585015012485938593e-2,  
a[20,1]=.9512362970482876694746379758949735521669033789834754257582261243292060560082303478949,  
a[20,2]=0.,  
a[20,3]=0.,  
a[20,4]=0.,  
a[20,5]=.2170506321979584863178462569531599428759163537577341676846566744497954872579456743665,  
a[20,6]=.1374557920759667598129078018350481905944439909394085308429177535450188051849310154998e-1,  
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a[20,8]=0.,  
a[20,9]=.1522816967364144471366046970407471319214864326994221120996167853723749050279525002079,  
a[20,10]=-3.377410183575998408023007931339980043546434244575396676700801526788878440490902531919,  
a[20,11]=-1.928259816339957815349491992868244004693531106307879821211327084555949417498597974185e-1,  
a[20,12]=-3.682592696968668099324090155354996035763121207468888802018816928929650528231675747053,  
a[20,13]=3.161978704069820635415335284196838540183520803428870023313121375688872348961613906410,  
a[20,14]=-3.704625221068852907169918560220511254779434822840805691773856863151016773010882921618,  
a[20,15]=-5.149742003654404349964344566981279849411686164743168710203142306658077550892325779417e-1,  
a[20,16]=-8.296255321201529467870435417928484166593826752027206775365542784220425091994969746694e-3,





