

## A modification of Hiroshi Ono's 17 stage order 10 Runge-Kutta scheme

See: A Runge-Kutta method of order 10 which minimizes truncation error, by Hiroshi Ono, Ibaraki University, The Japan Society for Industrial and Applied Mathematics, Vol. 13, No. 1, 2003, pp 35 - 44.

The scheme considered here can be constructed in the same manner as the scheme described in the preceding paper.

The nodes  $c_2$  and  $c_3$  (and consequently, by symmetry, also  $c_{15}$  and  $c_{16}$ ) along with the linking coefficient  $a_{14,10}$  are chosen so as to minimize the principal error norm subject to the condition that certain principal error terms are effectively zero.

The weights  $b_2$ ,  $b_3$  and  $b_7$  are chosen so as to minimize the 2-norm of the linking coefficients.

$$c_2 = c_{16} \simeq 0.3094568491778151756849284896200164891066,$$

$$c_3 = c_{15} \simeq 0.5223975466532662775097661019869049310092,$$

$$c_{17} = 1,$$

$$b_2 = -\frac{65}{3023}, b_3 = -\frac{45}{776}, b_4 = 0, b_5 = 0, b_6 = -\frac{16}{261}, b_7 = \frac{98}{617}, b_8 = 0, b_{13} = -\frac{98}{617}, b_{14} = \frac{16}{261}, b_{15} = \frac{45}{776}, b_{16} = \frac{65}{3023}.$$

The nodes:

$$c_9 = \frac{1}{2} - \frac{\sqrt{147 - 42\sqrt{7}}}{42}, c_{10} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42}, c_{11} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42}, c_{12} = \frac{1}{2} - \frac{\sqrt{147 + 42\sqrt{7}}}{42}$$

are the zeros of the derivative  $P'_5(x) = \frac{d}{dx}P_5(x)$  of the **Legendre polynomial**  $P_5(x)$  mapped linearly from the interval  $[-1, 1]$  to the interval  $[0, 1]$ .

They provide nodes for **Gauss-Lobatto integration** on the interval  $[0, 1]$ .

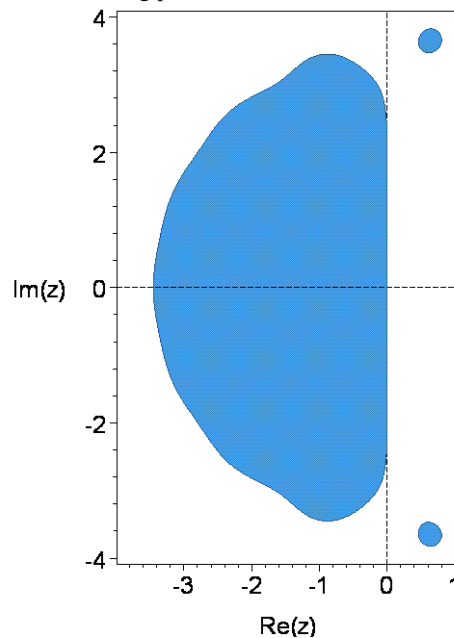
These values give rise to the following weights:

$$b_1 = \frac{1}{30}, b_9 = \frac{7}{30} + \frac{\sqrt{7}}{60}, b_{10} = \frac{7}{30} - \frac{\sqrt{7}}{60}, b_{11} = \frac{7}{30} + \frac{\sqrt{7}}{60}, b_{12} = \frac{7}{30} - \frac{\sqrt{7}}{60}, b_{17} = \frac{1}{30}.$$

The principal error norm, that is, the 2-norm of the principal error terms is:  $0.1137755077 \times 10^{-5}$ .

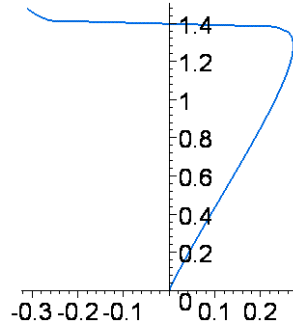
The maximum magnitude of the linking coefficients is: 1.300634802.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is  $[-3.4516, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval  $[0, 1.3902]$ .

The coefficients correct to 85 digits are as follows.

$c[2]=.3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,$   
 $c[3]=.5223975466532662775097661019869049310092389164141770790418748286187297481465201448588,$   
 $c[4]=.7835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[5]=.1815487914533370428642246610026788258089132736610314246989249804246268350054590436950,$   
 $c[6]=.2835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[7]=.7495559371815607246589400289542419288225215961442765650684876607643108166821502179989,$   
 $c[8]=.9663928215534496563098870275813343817004114475882666981126553255440139786693503644651e-1,$   
 $c[9]=.3573842417596774518429245029795604640404982636367873040901247917361510345429002009092,$   
 $c[10]=.8825276619647323464255014869796690751828678442680521196637911779185276585194132570617,$   
 $c[11]=.6426157582403225481570754970204395359595017363632126959098752082638489654570997990908,$   
 $c[12]=.1174723380352676535744985130203309248171321557319478803362088220814723414805867429383,$   
 $c[13]=.7495559371815607246589400289542419288225215961442765650684876607643108166821502179989,$   
 $c[14]=.2835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[15]=.5223975466532662775097661019869049310092389164141770790418748286187297481465201448588,$   
 $c[16]=.3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,$   
 $c[17]=1.,$

$a[2,1]=.3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,$   
 $a[3,1]=.8146499422405421895348275757224144991095476228935448377842649886254577098750110386134e-1,$   
 $a[3,2]=.4409325524292120585562833444146634810982841541248225952634483297561839771590190409975,$   
 $a[4,1]=.1958990799949748540661622882450893491284645936553164046407030607320236555549450543221,$   
 $a[4,2]=0.,$   
 $a[4,3]=.5876972399849245621984868647352680473853937809659492139221091821960709666648351629662,$   
 $a[5,1]=.1338434077333726797612927550600215619308683574208005562332042268793977361109095499660,$   
 $a[5,2]=0.,$   
 $a[5,3]=.8002250937438914898822994768838233528934759840402614534963991696250646186270445102172e-1,$   
 $a[5,4]=-.3231712565442478588529804174572507141130268216379527688391916341727736296815495729272e-1,$   
 $a[6,1]=.6421864439953820381246770491654875699805111189875475511565817021992178914940010906278e-1,$   
 $a[6,2]=0.,$   
 $a[6,3]=0.,$   
 $a[6,4]=.6406213153267746009903466912189352047987483920506291540859503005106411667890976371058e-3,$   
 $a[6,5]=.2187370542650344378511911013725897043110085143304602342930681224076621919035910105883,$   
 $a[7,1]=.2420381551105384778203242927555309371894325373540050621382918388920600526361289531724,$   
 $a[7,2]=0.,$   
 $a[7,3]=0.,$   
 $a[7,4]=.1190554466184020006132670816934739694238232918082467811740012896040358381992759219794,$   
 $a[7,5]=-.7590497680009786397812358427727688281617163202031917761669661091382604244380567961857,$   
 $a[7,6]=1.147512103453598886006584497278005850370982087185216497923160641406475350284802139033,$   
 $a[8,1]=.5712729213919824990099054363464512741497268175683453364621864638830432347535137406071e-1,$   
 $a[8,2]=0.,$   
 $a[8,3]=0.,$   
 $a[8,4]=0.,$

a[8,5]=.6686209040787335291342307899118079920053639602197019719266807767067619825747668578248e-1,  
a[8,6]=-.2796653051083541645116265166186605018009245768813918367155968045750214250894311926797e-1,  
a[8,7]=.6164301191087792677377317941735617346245246681611226439384889529230186430500958712855e-3,  
a[9,1]=.2445101401855880571753627785554865893686396081496046173624335026460692085807891247327e-1,  
a[9,2]=0.,  
a[9,3]=0.,  
a[9,4]=0.,  
a[9,5]=0.,  
a[9,6]=.1688298645137841333707208797636966823500202854631427117733126115029606114608148643875,  
a[9,7]=.1889806727065401202343581991642697008729001360215967078622437565407887268685688503991e-3,  
a[9,8]=.1639143825546279726344329871611508530527411172226625338727065862120427134971378551980,  
a[10,1]=.8532050922208605473377167896208955128734985134597789486440010978395195002785891096780e-1,  
a[10,2]=0.,  
a[10,3]=0.,  
a[10,4]=0.,  
a[10,5]=0.,  
a[10,6]=.1912892950915524975527582973719338867571795304811851440065568556880250454695479118841,  
a[10,7]=.3221310933502097348097535620317968621036380895623898017850624942556633703168517598378,  
a[10,8]=.2952154564086926983217592218598859494341086281756719489251270900624562476107391472900e-1,  
a[10,9]=.2542652186600147894970420264278601800912895100609320841152590091846416679440807596431,  
a[11,1]=-.2091962347396733971269867986546399595628499331654949353779957920786742089057287047774e-1,  
a[11,2]=0.,  
a[11,3]=0.,  
a[11,4]=0.,  
a[11,5]=0.,  
a[11,6]=-.4168966193636712125502529783669766021869578684730642430487665743912758872320559326790,  
a[11,7]=.7372903046439388959381121015470669267923184973406414808007248392664357406019556563115e-1,  
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a[11,9]=.7113831479391229536642577860785379988822857965017501665736029452476155732496203745297,  
a[11,10]=-.1696526126751994230123410831188754423111502568440552075893733327131966448195365459829e-1,  
a[12,1]=.2716351381181204031758152792191481577347446235693571825724929546586370958249852095032e-1,  
a[12,2]=0.,  
a[12,3]=0.,  
a[12,4]=0.,  
a[12,5]=0.,  
a[12,6]=-.6014118553563215704872758506419435407708235426383072805081732939868037511415358705603e-1,  
a[12,7]=-.1646130006362208258985530288999627575106856332675190625926233406310000849877034173647e-1,  
a[12,8]=.1130293781218100781634569414413965051732214644816689961068706236561081677521431133056,  
a[12,9]=.3627315404192112942662153437079909323284806895209096127449834350143039604870650540792e-1,  
a[12,10]=.4552294211582259763256738285372565994513363342756169453713303622133018218651195012604e-2,  
a[12,11]=.1305648344739638554216465895503857447122571418907866955395691929771743349151133705433e-1,  
a[13,1]=.1951224868957161494780169992510226643999073349014659857551581851236345024264850863153,  
a[13,2]=0.,  
a[13,3]=0.,  
a[13,4]=.1190554466184020006132670816934739694238232918082467811740012896040358381992759219794,  
a[13,5]=-.7590497680009786397812358427727688281617163202031917761669661091382604244380567961857,  
a[13,6]=.6941283840763331211943397035340831583782758902776183065835771589533250465130583624113,  
a[13,7]=.3977616143743016745236067888270644175689258547150351617511336547080963858868480538398,  
a[13,8]=.5680872909939944576037387913068410728867565697603808183011160497915901812968425997982,  
a[13,9]=.6044991449390174165625578226924885440940171194393334918312042732667501555440411674003,  
a[13,10]=-.8434505394085223900292603161981218926402706009966497845438498803643325878792356207118e-1,  
a[13,11]=-.4765838843827934583946936459945399798312546582126466893413258341183806655177078167106,  
a[13,12]=-.5091197243915797581377316379636109006721864262423005363650260193900469444407127987780,  
a[14,1]=.2027112243516135618798050718851840181199688439585160801787859203098020766329756164895,  
a[14,2]=0.,  
a[14,3]=0.,  
a[14,4]=.6406213153267746009903466912189352047987483920506291540859503005106411667890976371058e-3,  
a[14,5]=.2187370542650344378511911013725897043110085143304602342930681224076621919035910105883,

a[14,6]=- .2294480038501149350594512029689558547221621361559026845872482570339784200074296376999e-1,  
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a[14,8]=- 1.300634802218971811356036148941107828864676555563945435897417065124685413028522330267,  
a[14,9]=- .2080958440765482447813852317113504162126648093490294175145828608796379725779979383549e-1,  
a[14,10]=- .2567568e-1,  
a[14,11]=- .7790406294519129769264191766659722925425113094523349214774956853807024219706471789421e-1,  
a[14,12]= 1.216876068149946975731775374273677351438337315265486771994531293431667214425964027673,  
a[14,13]= 1.475419710499775337969189396279759282701683037483783111258518707348271401286912471960,  
a[15,1]= .8146499422405421895348275757224144991095476228935448377842649886254577098750110386134e-1,  
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a[15,3]= 0.,  
a[15,4]= 0.,  
a[15,5]= 0.,  
a[15,6]=- .5679992395357323657508375553381789632748361699043656500590116980718248512464373449649,  
a[15,7]= .4138086298267394454063188438358776530567576801014340642463931055705685884769862214481,  
a[15,8]= 0.,  
a[15,9]= 0.,  
a[15,10]= 0.,  
a[15,11]= 0.,  
a[15,12]= 0.,  
a[15,13]=- .4138086298267394454063188438358776530567576801014340642463931055705685884769862214481,  
a[15,14]= .5679992395357323657508375553381789632748361699043656500590116980718248512464373449649,  
a[16,1]= .3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,  
a[16,2]= 0.,  
a[16,3]=- .4454386151049480716413470986184849584797024707326341527201681579183670881387716047240,  
a[16,4]= 0.,  
a[16,5]= 0.,  
a[16,6]= 0.,  
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a[16,9]= 0.,  
a[16,10]= 0.,  
a[16,11]= 0.,  
a[16,12]= 0.,  
a[16,13]= 0.,  
a[16,14]= 0.,  
a[16,15]= .4454386151049480716413470986184849584797024707326341527201681579183670881387716047240,  
a[17,1]=- .1462288863880059452226171233611639566030087808828980546439453482155763115642661362247,  
a[17,2]=- .4454380231899637470772045799672404387171925919380573123604080215230100465544208458115,  
a[17,3]=- .5435483372553891966406530260514961720167972270391349446681960101862780041823062579313,  
a[17,4]= 0.,  
a[17,5]= 0.,  
a[17,6]=- 1.127468787604960875417657055312981778492070347271938051827857633912761661789481819991,  
a[17,7]= .1193021198373799489161405309336442296900119097015467262915622543796790151932607361110,  
a[17,8]= .326275121518117751075674729809746957897026021782875904586649311113367600130062449909,  
a[17,9]= .6969391247111585945021065788552067434519039562051650648708879227536477227415375654279,  
a[17,10]= .4275770246707114093918415765910329263586994174613194844537030947716978246924634777574,  
a[17,11]= .7726948183013579653507012049457348309311647990390840747357791085452526708668004197391,  
a[17,12]= .3463349145514361344476314894313151367627747683627512015113699266158080811657579128060,  
a[17,13]=- .7448064519052872054508797925912213444728886985950256058113384094154773784594093351967,  
a[17,14]= .3293810023080922224070578606986862544763869541971192558331897733663932771403309345801,  
a[17,15]= .5435483372553891966406530260514961720167972270391349446681960101862780041823062579313,  
a[17,16]= .4454380231899637470772045799672404387171925919380573123604080215230100465544208458115,

b[1]= .333e-1,  
b[2]=- .2150181938471716837578564340059543499834601389348329474032418127687727423089646046973e-1,  
b[3]=- .5798969072164948453608247422680412371134020618556701030927835051546391752577319587629e-1,  
b[4]= 0.,

