

**A modification of Hiroshi Ono's order 10 Runge-Kutta scheme with an order 8 embedded scheme**

See: A Runge-Kutta method of order 10 which minimizes truncation error, by Hiroshi Ono, Ibaraki University, The Japan Society for Industrial and Applied Mathematics, Vol. 13, No. 1, 2003, pp 35 - 44.

**Note:** The embedded order 8 scheme requires 3 stages to be added to Hiroshi Ono's 17 stage scheme.  
 The order 8 embedded scheme satisfies 116 of the 286 order 8 principal error conditions.

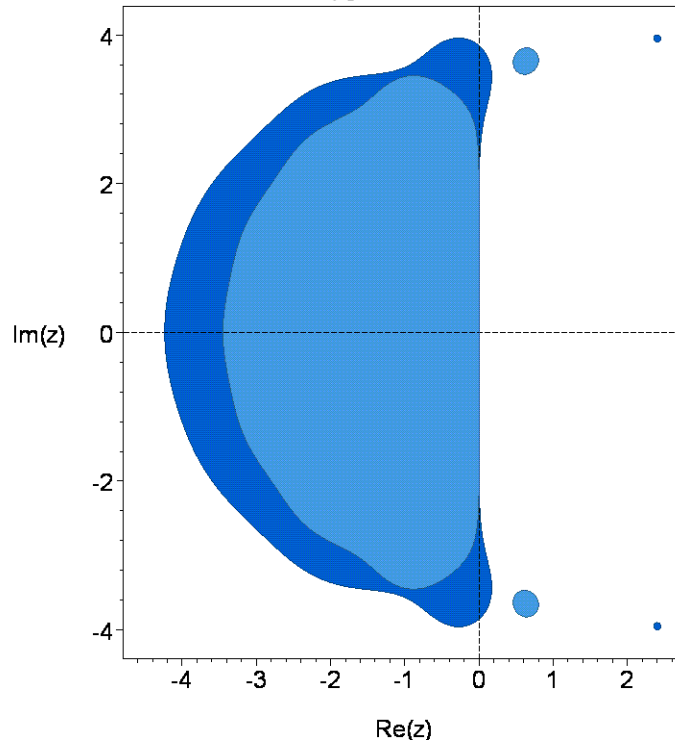
The principal error norm of the order 10 scheme, that is, the 2-norm of the principal error terms is:  $0.1137755077 \times 10^{(-5)}$ .

The principal error norm of the embedded order 8 scheme is:  $0.5517017016 \times 10^{(-5)}$ .

The maximum magnitude of the linking coefficients is 3.449474867.

The 2-norm of the linking coefficients is 7.048786321.

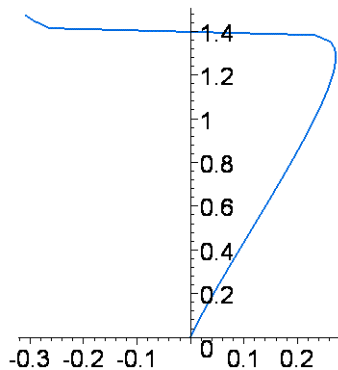
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 8 scheme appears in the darker shade.

The real stability intervals of the order 10 and 8 schemes are respectively  $[-3.4516, 0]$  and  $[-4.2358, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval  $[0, 1.3902]$ .

The coefficients correct to 85 digits are as follows.

$c[2]=.3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,$   
 $c[3]=.5223975466532662775097661019869049310092389164141770790418748286187297481465201448588,$   
 $c[4]=.7835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[5]=.1815487914533370428642246610026788258089132736610314246989249804246268350054590436950,$   
 $c[6]=.2835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[7]=.7495559371815607246589400289542419288225215961442765650684876607643108166821502179989,$   
 $c[8]=.9663928215534496563098870275813343817004114475882666981126553255440139786693503644651e-1,$   
 $c[9]=.3573842417596774518429245029795604640404982636367873040901247917361510345429002009092,$   
 $c[10]=.8825276619647323464255014869796690751828678442680521196637911779185276585194132570617,$   
 $c[11]=.6426157582403225481570754970204395359595017363632126959098752082638489654570997990908,$   
 $c[12]=.1174723380352676535744985130203309248171321557319478803362088220814723414805867429383,$   
 $c[13]=.7495559371815607246589400289542419288225215961442765650684876607643108166821502179989,$   
 $c[14]=.2835963199798994162646491529803573965138583746212656185628122429280946222197802172882,$   
 $c[15]=.5223975466532662775097661019869049310092389164141770790418748286187297481465201448588,$   
 $c[16]=.3094568491778151756849284896200164891066291254211552537099931030440721175723004015958,$   
 $c[17]=1.,$   
 $c[18]=.3598901098901098901098901098901098901098901098901098901098901098901098901098901098901,$   
 $c[19]=.6430062630480167014613778705636743215031315240083507306889352818371607515657620041754,$   
 $c[20]=1.,$

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 $a[9,3]=0.,$   
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 $a[9,6]=.1688298645137841333707208797636966823500202854631427117733126115029606114608148643875,$   
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a[15,5]=0.,  
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a[15,10]=0.,  
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a[18,13]=0.,  
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a[18,16]=0.,  
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b\*[1]=.3350081881531992998954114344916870405487380000412887152465537770001367461780701209303e-1,  
b\*[2]=0.,  
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b\*[4]=0.,  
b\*[5]=0.,  
b\*[6]=.4290552442894369722055848233967368632203608750773861081635140926793471935201135822486e-2,  
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b\*[8]=-.2269995994124716250500734410468687408198691414073975163573240753104553344905861930832e-2,  
b\*[9]=.9401709401709401709401709401709401709401709401709401709401709401709401709401709401709e-1,  
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b\*[13]=0.,  
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b\*[19]=.2236255786266114717067168337721999557944781461211775191504366233476214957258751684360,  
b\*[20]=.3337027376600852807054337560731601671291228525063926206104788641102318303617712826821e-1

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