

Curtis' 18 stage order 10 Runge-Kutta scheme

See: High-order Explicit Runge-Kutta Formulae, Their uses, and Limitations, A.R.Curtis, J. Inst. Maths Applics (1975) **16**, 35-55.

The nodes:

$$c_{12} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42}, \quad c_{13} = \frac{1}{2} - \frac{\sqrt{147 + 42\sqrt{7}}}{42}, \quad c_{14} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42}, \quad c_{15} = \frac{1}{2} - \frac{\sqrt{147 - 42\sqrt{7}}}{42},$$

$$c_{16} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42}, \quad c_{17} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42}.$$

are the zeros of the derivative $P'_5(x) = \frac{d}{dx}P_5(x)$ of the **Legendre polynomial** $P_5(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$. They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

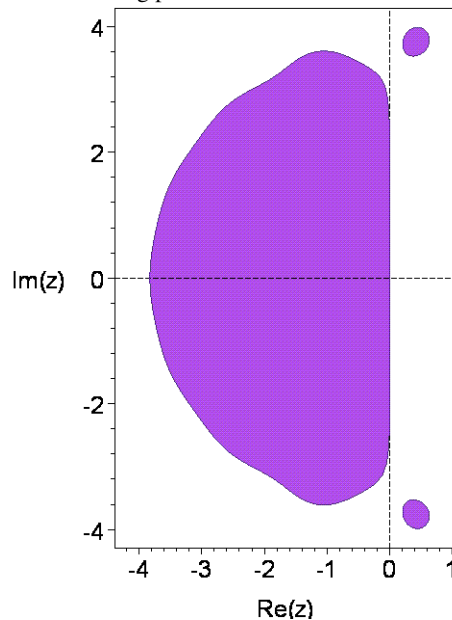
These values give rise to the following weights:

$$b_1 = \frac{1}{30}, \quad b_{12} = \frac{7}{60} + \frac{\sqrt{7}}{120}, \quad b_{13} = \frac{7}{30} - \frac{\sqrt{7}}{60}, \quad b_{14} = \frac{7}{60} - \frac{\sqrt{7}}{120}, \quad b_{15} = \frac{7}{30} + \frac{\sqrt{7}}{60}, \quad b_{16} = \frac{7}{60} + \frac{\sqrt{7}}{120}, \quad b_{17} = \frac{7}{60} - \frac{\sqrt{7}}{120}, \quad b_{18} = \frac{1}{30}.$$

The remaining weights are zero, that is, $b_i = 0, i = 2 \dots 11$.

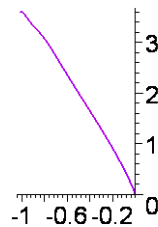
The principal error norm, that is, the 2-norm of the principal error terms is: $0.3502626170 \times 10^{(-5)}$.
The maximum magnitude of the linking coefficients is: 5.472417145.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is $[-3.8269, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis only at the origin.

The coefficients correct to 85 digits are as follows.

$c[2]=.1452518960316150517617548528770033320314511251329947060838468741983976455607179673401,$
 $c[3]=.1452518960316150517617548528770033320314511251329947060838468741983976455607179673401,$
 $c[4]=.2178778440474225776426322793155049980471766876994920591257703112975964683410769510101,$
 $c[5]=.5446946101185564441065806982887624951179417192487301478144257782439911708526923775252,$
 $c[6]=.6536335321422677329278968379465149941415300630984761773773109338927894050232308530303,$
 $c[7]=.2746594919905254008808021630247618520892150865127407293922085868737635475402543533498,$
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 $c[12]=.6426157582403225481570754970204395359595017363632126959098752082638489654570997990908,$
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 $c[18]=1.,$

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 $a[9,4]=0.,$
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