Construction of 22 stage combined order 9 and 10 Runge-Kutta schemes

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A method for the constuction of 22 stage combined 9 and 10 Runge-Kutta schemes is described here. The embedded order 10 scheme is provided with the aim of it being used for error control in the standard manner. The method of construction of these schemes is motivated by a 21 stage 10(9) scheme due to Tom Baker at the University of Teeside. The inclusion of an extra step allows for greater variability in possible characteristics of the order 10 scheme including the possibility of arranging for certain principal error terms to be zero.

When the construction reaches the stage of requiring equations derived from the order conditions, the order conditions used at each step were selected from those that remained to be satisfied after the preceding step. Details of how this was achieved can be seen by viewing Maple worksheets which are available online.

We first indicate how to construct a 21 stage order 10 Runge-Kutta scheme.
Step 1:
We specify the following stage-orders for stages 3 to 16.

$$\begin{bmatrix} \textit{stage} & | & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \textit{stage-order} & | & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

Thus we have the following stage-order conditions involving the nodes c_i and linking coefficients $a_{i,j}$.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \text{ for } i = 3..16, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j = \frac{c_i^2}{2} \text{ for } i = 3..16, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^2 = \frac{c_i^3}{3} \text{ for } i = 3..16,$$

$$\sum_{j=2}^{i-1} a_{i,j}\, c_j^3 = \frac{c_i^4}{4} \text{ for } i = 6..16, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^4 = \frac{c_i^5}{5} \text{ for } i = 9..16, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^5 = \frac{c_i^6}{6} \text{ for } i = 12..16.$$

We also have zero linking coefficients as indicated by the following tableau.

$$\begin{bmatrix}
c_2 & a_{2,1} \\
c_3 & a_{3,1} & a_{3,2} \\
c_4 & a_{4,1} & 0 & a_{4,3} \\
c_5 & a_{5,1} & 0 & a_{5,3} & a_{5,4} \\
c_6 & a_{6,1} & 0 & 0 & a_{6,4} & a_{6,5} \\
c_7 & a_{7,1} & 0 & 0 & 0 & a_{7,5} & a_{7,6} \\
c_8 & a_{8,1} & 0 & 0 & 0 & a_{8,5} & a_{8,6} & a_{8,7} \\
c_9 & a_{9,1} & 0 & 0 & 0 & 0 & a_{9,6} & a_{9,7} & a_{9,8} \\
c_{10} & a_{10,1} & 0 & 0 & 0 & 0 & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} \\
c_{11} & a_{11,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{11,8} & a_{11,9} & a_{11,10} \\
c_{12} & a_{12,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12,9} & a_{12,10} & a_{12,11} \\
c_{13} & a_{13,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{13,9} & a_{13,10} & a_{13,11} & a_{13,12} \\
c_{14} & a_{14,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{14,9} & a_{14,10} & a_{14,11} & a_{14,12} & a_{14,13} \\
c_{15} & a_{15,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{15,9} & a_{15,10} & a_{15,11} & a_{15,12} & a_{15,13} & a_{15,14} \\
c_{16} & a_{16,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{16,9} & a_{16,10} & a_{16,11} & a_{16,12} & a_{16,13} & a_{16,14} & a_{16,15}
\end{bmatrix}$$

We can use this information to obtain certain relations involving the nodes c_2 to c_{16}.

$$\int_0^{c_3} x\,(x - c_2)\,dx = 0 \quad \text{or} \quad c_2 = \frac{2}{3}\,c_3 \qquad \text{------------- (i)}$$

$$\int_0^{c_4} x\,(x-c_3)\,dx = 0 \quad \text{or} \quad c_3 = \frac{2}{3}\,c_4 \quad \text{------------- (ii)}$$

$$\int_0^{c_6} x\,(x-c_4)\,(x-c_5)\,dx = 0 \quad \text{or} \quad c_4 = \frac{c_6\,(3\,c_6 - 4\,c_5)}{2\,(2\,c_6 - 3\,c_5)} \quad \text{------------- (iii)}$$

$$\int_0^{c_9} x\,(x-c_6)\,(x-c_7)\,(x-c_8)\,dx = 0 \quad \text{or} \quad c_6 = \frac{(20\,c_8\,c_7 - 15\,c_9\,c_8 + 12\,c_9^{\,2} - 15\,c_9\,c_7)\,c_9}{5\,(6\,c_8\,c_7 - 4\,c_9\,c_8 + 3\,c_9^{\,2} - 4\,c_9\,c_7)} \quad \text{------------- (iv)}$$

$$\int_0^{c_{11}} x\,(x-c_8)\,(x-c_9)\,(x-c_{10})\,dx = 0 \quad \text{or} \quad c_8 = \frac{(15\,c_9\,c_{11} - 20\,c_9\,c_{10} - 12\,c_{11}^{\,2} + 15\,c_{10}\,c_{11})\,c_{11}}{5\,(4\,c_9\,c_{11} - 6\,c_9\,c_{10} - 3\,c_{11}^{\,2} + 4\,c_{10}\,c_{11})} \quad \text{------------- (v)}$$

$$\int_0^{c_{12}} x\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,dx = 0 \quad \text{------------- (vi)}$$

$$\int_0^{c_{12}} x^2\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,dx = 0 \quad \text{------------- (vii)}$$

$$\int_0^{c_{13}} x\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,(x-c_{12})\,dx = 0 \quad \text{------------- (viii)}$$

By introducing the relation $c_{11} = \frac{2}{3}\,c_{13}$ the equations (vi), (vii) and (viii) allow the following relations to be obtained.

$$c_9 = \frac{98 - \sqrt{10} + \sqrt{3694 - 212\sqrt{10}}}{156}\,c_{13}, \qquad c_{10} = \frac{98 - \sqrt{10} - \sqrt{3694 - 212\sqrt{10}}}{156}\,c_{13}, \qquad c_{12} = \frac{40 - 2\sqrt{10}}{39}\,c_{13}.$$

Specifying values for the nodes $c_7,\ c_{13},\ c_{14},\ c_{15},\ c_{16},$ and the zero linking coefficients $a_{15,\,14},\ \ a_{16,\,14},\ a_{16,\,15}$ along with the assumption that certain linking coefficients are zero as given in the preceding table
allows all the nodes and linking coefficients in stages 2 to 16 to be calculated.

Step 2:
We specify the remaining nodes $c_{17},\ c_{18},\ c_{19},\ c_{20},\ c_{21} = 1$ and the weight b_{21}.
We also require that $b_i = 0,\ i = 2\,..\,11$. Then we can obtain all the remaining weights by using the order 10 quadrature conditions:

$$\sum_{i=1}^{21} b_i = 1, \qquad \sum_{i=1}^{21} b_i\,c_i^{\,k} = \frac{1}{k+1}, \quad k = 1\,..\,9.$$

Step 3:
The stages 17 to 21 all have stage-order 6. Thus the following conditions hold.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \ \text{ for } i = 17\,..\,21, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j^{\,k} = \frac{c_i^{\,(k+1)}}{k+1} \ \text{ for } i = 17\,..\,21,\ k = 1\,..\,5.$$

After specifying the zero linking coefficients $a_{i,j} = 0$ for $j = 2\,..\,8,\ i = 17\,..\,21$, these equations provide linear relations among the linking coefficients in stages 17 to 21.
We can obtain linear expressions for 30 of the linking coefficients with the following 25 linking coefficients remaining as parameters.

$$a_{17,\,i},\ i = 9, 13, 16, \quad a_{18,\,i},\ i = 9, 13, 16, 17, \quad a_{19,\,i},\ i = 9, 12, 13, 16, 17,$$
$$a_{20,\,i},\ i = 9, 12, 13, 16, 17, 19, \quad a_{21,\,i},\ i = 9, 12, 13, 16, 17, 19, 20.$$

Step 4:
A system of equations arising from the 6 column simplifying conditions

$$\sum_{i=j+1}^{21} b_i\, a_{i,j} = b_j\,(1-c_j), \quad j = 9, 13, 16, 17, 19, 20.$$

can be solved to give linear expressions for $a_{18,9},\ a_{18,13},\ a_{18,16},\ a_{18,17},\ a_{21,19},\ a_{21,20}$.

By means of substitution we obtain linear expressions for 36 linking coefficients with the following 19 linking coefficients remaining as parameters.

$$a_{17,i},\ i = 9, 13, 16, \quad a_{19,i},\ i = 9, 12, 13, 16, 17, \quad a_{20,i},\ i = 9, 12, 13, 16, 17, 19, \quad a_{21,i},\ i = 9, 12, 13, 16, 17.$$

Step 5:
A system of equations arising from the 8 order conditions

$$\sum_{i=3}^{21} b_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,8}\right) = \frac{1}{90}, \qquad \sum_{i=3}^{21} b_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,7}\right) = \frac{1}{80}, \qquad \sum_{i=3}^{21} b_i\, c_i^{\,2} \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,6}\right) = \frac{1}{70},$$

$$\sum_{i=4}^{21} b_i\, c_i^{\,2} \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5}\right)\right) = \frac{1}{420}, \qquad \sum_{i=4}^{21} b_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5}\right)\right) = \frac{1}{480},$$

$$\sum_{i=5}^{21} b_i\, c_i^{\,2} \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,4}\right)\right)\right) = \frac{1}{2100},$$

$$\sum_{i=5}^{21} b_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j}\, c_j \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,4}\right)\right)\right) = \frac{1}{2400},$$

$$\sum_{i=6}^{21} b_i\, c_i^{\,2} \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,3}\right)\right)\right)\right) = \frac{1}{8400}.$$

can be solved to give linear expressions for $a_{17,9},\ a_{17,13},\ a_{17,16},\ a_{19,9},\ a_{19,12},\ a_{19,13},\ a_{19,16},\ a_{19,17}$.

By means of substitution we obtain linear expressions for 44 linking coefficients with the following 11 linking coefficients remaining as parameters.

$$a_{20,i},\ i = 9, 12, 13, 16, 17, 19, \quad a_{21,i},\ i = 9, 12, 13, 16, 17.$$

Step 6:
A system of equations arising from the 2 order conditions

$$\sum_{i=4}^{21} b_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,6}\right)\right) = \frac{1}{630}, \qquad \sum_{i=5}^{21} b_i \left(\sum_{j=4}^{i-1} a_{i,j}\, c_j \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,5}\right)\right)\right) = \frac{1}{3780}$$

can be solved to give linear expressions for $a_{21,12}$ and $a_{20,19}$.

By means of substitution we obtain linear expressions for 46 linking coefficients in terms of the 9 linking coefficients

$$a_{20,i},\ i = 9, 12, 13, 16, 17, \quad a_{21,i},\ i = 9, 13, 16, 17.$$

Step 7:
We now specify values for 5 of the linking coefficients in rows 20 and 21, namely $a_{20,9},\ a_{20,12},\ a_{20,13},\ a_{21,9}$ and $a_{21,13}$.
By means of substitution we obtain linear expressions for 51 linking coefficients in terms of the 4 linking coefficients

$$a_{20,16},\ a_{20,17},\ a_{21,16},\ a_{21,17}.$$

A system of equations in the 4 variable linking coefficients can be constructed from the 4 order conditions

$$\sum_{i=5}^{21} b_i \left(\sum_{j=4}^{i-1} a_{i,j}\left(\sum_{k=3}^{j-1} a_{j,k}\left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,6}\right)\right)\right) = \frac{1}{5040},$$

$$\sum_{i=6}^{21} b_i \left(\sum_{j=5}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=3}^{k-1} a_{k,l}\left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,5}\right)\right)\right)\right) = \frac{1}{30240},$$

$$\sum_{i=6}^{21} b_i\, c_i \left(\sum_{j=5}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=3}^{k-1} a_{k,l}\left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,4}\right)\right)\right)\right) = \frac{1}{16800},$$

$$\sum_{i=7}^{21} b_i\, c_i \left(\sum_{j=6}^{i-1} a_{i,j}\left(\sum_{k=5}^{j-1} a_{j,k}\left(\sum_{l=4}^{k-1} a_{k,l}\left(\sum_{m=3}^{l-1} a_{l,m}\left(\sum_{n=2}^{m-1} a_{m,n}\, c_n^{\,3}\right)\right)\right)\right)\right) = \frac{1}{67200}.$$

The resulting 4 equations all have degree 2 in the 4 variables $a_{20,16}$, $a_{20,17}$, $a_{21,16}$, $a_{21,17}$ and so would be difficult to solve analytically.
However, the system can be solved by using the multidimensional version of Newton's method if initial values are provided.
In many cases it is sufficient to take the initial value for each of the 4 variables to be zero.
A given order 11 scheme is determined by specifying values for the 19 coefficients

$$c_7,\ c_{13},\ c_{14},\ c_{15},\ c_{16},\ c_{17},\ c_{18},\ c_{19},\ c_{20},\ a_{15,14},\ a_{16,14},\ a_{16,15},\ b_{21},\ a_{20,9},\ a_{20,12},\ a_{20,13},\ a_{21,9},\ a_{21,13}.$$

In searching for order 10 schemes with reasonable characterstics one may change the values of these 18 parameters incrementally and use the values obtained at a given stage for the 4 coefficients which occur as the variables in the system of non-linear equations to be solved in the last step of the construction of the scheme as starting values for Newton's method in the determination of the coefficients for the next scheme.

Once an order 10 scheme has been constructed, an embedded scheme order 9 scheme can be obtained by adding a 22th row of linking coefficients. We specify that $c_{22} = 1$, $a_{22,i} = 0$ for $i = 2\,..\,8$, $b^*_i = 0$ for $i = 2\,..\,11$. We set $b^*_{21} = 0$ and give a value for b^*_{22} which means that the scheme is essentially a 21 stage scheme.

A system of equations can be constructed using the order 9 quadrature conditions together with the row-sum condition for the additional 22th stage and the stage-order conditions that ensure that this stage has stage-order 6, that is,

$$\sum_{i=1}^{22} b^*_i = 1, \qquad \sum_{i=1}^{22} b^*_i\, c_i^{\,k} = \frac{1}{k+1} \ \text{ for } k = 1\,..\,8,$$

$$\sum_{j=1}^{21} a_{22,j} = c_{22}, \qquad \sum_{j=2}^{21} a_{22,j}\, c_j^{\,k} = \frac{1}{k+1}\, c_{22}^{\,(k+1)} \ \text{ for } k = 1\,..\,5.$$

In choosing a value for b^*_{22} we try to ensure that the stability region of the embedded order 9 scheme is compatible with that of the order 10 scheme. We make the order 10 scheme into a 22 stage scheme by specifying that $b_{22} = 0$.
The system of 15 equations can be solved to express the weights b^*_1 and b^*_i for i = 12 to 19 in terms of b^*_{20} .
Additionally, the linking coefficients $a_{22,1}$ and $a_{22,i}$, i = 9 to 13 are expressed as linear combinations of the linking coefficients $a_{22,i}$, i = 14 to 21.
The column simplifying conditions

$$\sum_{i=j+1}^{22} b^*_i\, a_{i,j} = b^*_j\,(1 - c_j), \ \ j = 14\,..\,21,$$

can now be used to determine the linking coefficients $a_{22,j}$ for $j = 14\,..\,21$ in terms of b^*_{20}.
This enables all the linking to be expressed in tems of b^*_{20}.

Finally, b^*_{20} can be determined by using the single order condition

$$\sum_{i=3}^{22} b^*_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,6} \right) = \frac{1}{63} .$$

The stage-orders of stages 3 to 22 of the combined scheme are as follows.

stage	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
stage-order	3	3	3	4	4	4	5	5	5	6	6	6	6	6	6	6	6	6	6	6

The principal error norm of an order 10 scheme constructed in the manner described can be calculated using 24 of the 1842 principal error terms. These error terms are given in an abreviated form as follows.

$$b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))) - \frac{1}{3991680}, \qquad b\,c\,(a\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{498960},$$

$$\frac{1}{2}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{443520}\right), \qquad \frac{1}{6}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c^3)))))) - \frac{1}{665280}\right),$$

$$\frac{1}{6}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))) - \frac{1}{55440}\right), \qquad \frac{1}{6}\left(b\,c\,(a\,c\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{83160}\right),$$

$$\frac{1}{12}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{73920}\right), \qquad \frac{1}{24}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,c^4))))) - \frac{1}{166320}\right),$$

$$\frac{1}{36}\left(b\,c^3\,(a\,(a\,(a\,(a\,c^3)))) - \frac{1}{9240}\right), \qquad \frac{1}{24}\left(b\,c\,(a\,c\,(a\,(a\,(a\,c^4)))) - \frac{1}{20790}\right),$$

$$\frac{1}{48}\left(b\,c^2\,(a\,(a\,(a\,(a\,c^4)))) - \frac{1}{18480}\right), \qquad \frac{1}{120}\left(b\,c\,(a\,(a\,(a\,(a\,c^5)))) - \frac{1}{33264}\right),$$

$$\frac{1}{144}\left(b\,c^3\,(a\,(a\,(a\,c^4))) - \frac{1}{2310}\right), \qquad \frac{1}{120}\left(b\,c\,(a\,c\,(a\,(a\,c^5))) - \frac{1}{4158}\right), \qquad \frac{1}{240}\left(b\,c^2\,(a\,(a\,(a\,c^5))) - \frac{1}{3696}\right),$$

$$\frac{1}{720}\left(b\,c\,(a\,(a\,(a\,c^6))) - \frac{1}{5544}\right), \qquad \frac{1}{720}\left(b\,c^3\,(a\,(a\,c^5)) - \frac{1}{462}\right), \qquad \frac{1}{720}\left(b\,c\,(a\,c\,(a\,c^6)) - \frac{1}{693}\right),$$

$$\frac{1}{1440}\left(b\,c^2\,(a\,(a\,c^6)) - \frac{1}{616}\right), \qquad \frac{1}{5040}\left(b\,c\,(a\,(a\,c^7)) - \frac{1}{792}\right), \qquad \frac{1}{4320}\left(b\,c^3\,(a\,c^6) - \frac{1}{77}\right),$$

$$\frac{1}{10080}\left(b\,c^2\,(a\,c^7) - \frac{1}{88}\right), \qquad \frac{1}{40320}\left(b\,c\,(a\,c^8) - \frac{1}{99}\right), \qquad \frac{1}{3628800}\left(b\,c^{10} - \frac{1}{11}\right).$$

For example, $\frac{1}{240}\left(b\,c^2\,(a\,(a\,(a\,c^5))) - \frac{1}{3696}\right)$ is an abreviation for

$$\frac{1}{240}\left(\left(\sum_{i=2}^{21} b_i\, c_i^{\,2}\left(\sum_{j=3}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=5}^{k-1} a_{k,l}\, c_l^{\,5}\right)\right)\right)\right) - \frac{1}{3696}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{24} d_i\, e_i^{\,2}}$, where e_i is the value of the ith error term above and d_i is the ith member of the sequence
2, 2, 6, 22, 30, 22, 66, 156, 330, 156, 468, 2052, 2340, 2052, 6156, 39432, 30780, 39432, 118296, 950112, 591480, 2850336, 26381700, 65759327412.

The principal error norm of an order 9 embedded scheme constructed in the manner described can be calculated using 12 of the 719 principal error terms. These error terms are given in an abreviated form as follows.

$$b{*}\,c\ (a\ (a\ (a\ (a\ (a\ (a\ (a\ c))))))) - \frac{1}{403200},\qquad \frac{1}{2}\left(b{*}\,c^2\ (a\ (a\ (a\ (a\ (a\ (a\ c)))))) - \frac{1}{50400}\right),$$

$$\frac{1}{6}\left(b{*}\,c\ (a\ (a\ (a\ (a\ (a\ c^3))))) - \frac{1}{67200}\right),\qquad \frac{1}{12}\left(b{*}\,c^2\ (a\ (a\ (a\ (a\ c^3)))) - \frac{1}{8400}\right),$$

$$\frac{1}{24}\left(b{*}\,c\ (a\ (a\ (a\ (c^4\ a)))) - \frac{1}{16800}\right),\qquad \frac{1}{48}\left(b{*}\,c^2\ (a\ (a\ (a\ c^4))) - \frac{1}{2100}\right),$$

$$\frac{1}{120}\left(b{*}\,c\ (a\ (a\ (a\ c^5))) - \frac{1}{3360}\right),\qquad \frac{1}{240}\left(b{*}\,c^2\ (a\ (a\ c^5)) - \frac{1}{420}\right),\qquad \frac{1}{720}\left(b{*}\,c\ (a\ (a\ c^6)) - \frac{1}{560}\right),$$

$$\frac{1}{1440}\left(b{*}\,c^2\ (a\ c^6) - \frac{1}{70}\right),\qquad \frac{1}{5040}\left(b{*}\,c\ (a\ c^7) - \frac{1}{80}\right),\qquad \frac{1}{362880}\left(b{*}\,c^9 - \frac{1}{10}\right).$$

For example, $\frac{1}{48}\left(b{*}\,c^2\ (a\ (a\ (a\ c^4))) - \frac{1}{2100}\right)$ is an abreviation for

$$\frac{1}{48}\left(\left(\sum_{i=2}^{22} b{*}_i\,c_i^2\left(\sum_{j=3}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=5}^{k-1} a_{k,l}\,c_l^4\right)\right)\right)\right) - \frac{1}{2100}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{12} \delta_i\, {e{*}_i}^2}$, where $e{*}_i$ is the value of the ith error term above and δ_i is the ith member of the sequence
2, 6, 22, 66, 156, 468, 2052, 6156, 39432, 118296, 950112, 1531572132.

The following sets of parameter values determine schemes with the given characteristics. A complete list of the values of the coefficients (correct to 85 digits) for the 9 schemes listed here are given in separate documents.

The first three schemes have progressively smaller principal error norms for the order 10 scheme.
1. **First scheme with a small principal error norm**.

$$c_7 = \frac{45}{368},\ c_{13} = \frac{359}{485},\ c_{14} = \frac{232}{593},\ c_{15} = \frac{603}{5674},\ c_{16} = \frac{897}{1031},\ c_{17} = \frac{775}{804},\ c_{18} = \frac{176}{545},\ c_{19} = \frac{873}{1384},\ c_{20} = \frac{297}{307},$$

$$a_{15,14} = -\frac{113}{722},\ a_{16,14} = \frac{126}{293},\ a_{16,15} = \frac{241}{522},\ b_{21} = -\frac{103}{890},$$

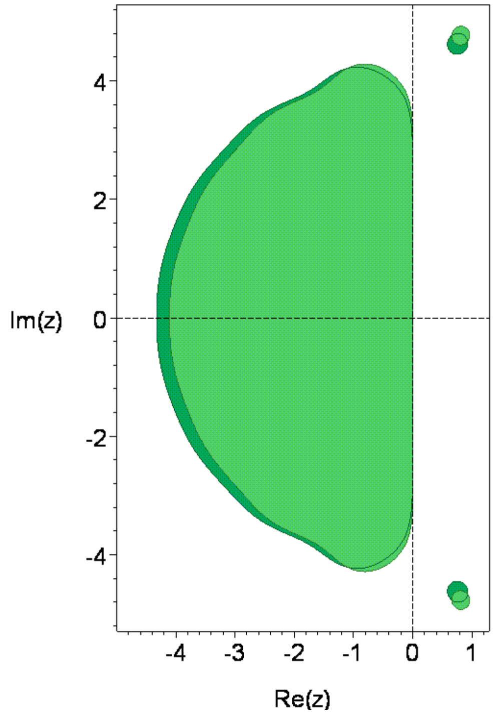
$$a_{20,9} = \frac{204}{785},\ a_{20,12} = \frac{315}{353},\ a_{20,13} = -\frac{231}{388},\ a_{21,9} = \frac{530}{349},\ a_{21,13} = \frac{77}{40}.$$

The principal error norm of the order 10 scheme is $0.56347046 \times 10^{(-7)}$ and the real stability interval is $[-4.10684, 0]$.

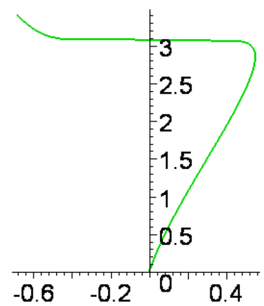
The weight that determines the order 9 embedded scheme is $b{*}_{22} = -\frac{5}{53}$.

The principal error norm of the order 9 scheme is $0.29344131 \times 10^{(-6)}$ and the real stability interval is $[-4.32463, 0]$.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 3.084705\, i]$.

2. **Second scheme with a small principal error norm**.

$$c_7 = \frac{19}{156},\ c_{13} = \frac{465}{628},\ c_{14} = \frac{343}{872},\ c_{15} = \frac{124}{1173},\ c_{16} = \frac{918}{1057},\ c_{17} = \frac{295}{306},\ c_{18} = \frac{185}{579},\ c_{19} = \frac{305}{483},\ c_{20} = \frac{622}{643},$$

$$a_{15,14} = -\frac{177}{1114},\ a_{16,14} = \frac{94}{211},\ a_{16,15} = \frac{181}{394},\ b_{21} = -\frac{106}{1069},$$

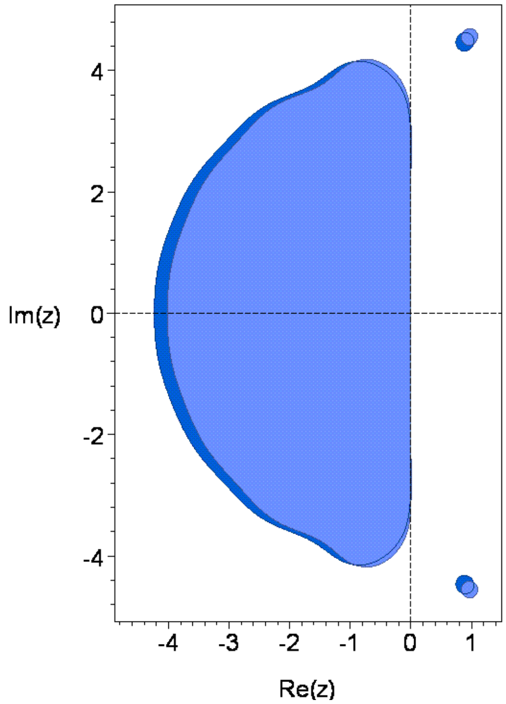
$$a_{20,9} = \frac{295}{1038},\ a_{20,12} = \frac{83}{94},\ a_{20,13} = -\frac{758}{937},\ a_{21,9} = \frac{99}{68},\ a_{21,13} = \frac{325}{127}.$$

The principal error norm of the order 10 scheme is $0.54230636 \times 10^{(-7)}$ and the real stability interval is $[-4.01873, 0]$.

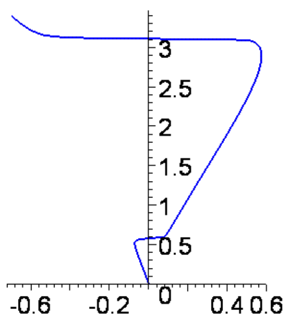
The weight that determines the order 9 embedded scheme is $b^*_{22} = -\frac{2}{25}$.

The principal error norm of the order 9 scheme is $0.28364680 \times 10^{(-6)}$ and the real stability interval is $[-4.23297, 0]$.
The maximum magnitude of the linking coefficients is 30.56350004 and the 2-norm of the linking coefficients is 70.35689161.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval [0.56600 i, 3.11891 i] as well as at the origin.

3. **Third scheme with a small principal error norm**.

$$c_7 = \frac{161}{1375},\ c_{13} = \frac{964}{1301},\ c_{14} = \frac{311}{781},\ c_{15} = \frac{185}{1766},\ c_{16} = \frac{608}{705},\ c_{17} = \frac{1071}{1111},\ c_{18} = \frac{200}{637},\ c_{19} = \frac{695}{1099},\ c_{20} = \frac{3641}{3764},$$

$$a_{15,\,14} = -\frac{18}{109},\ a_{16,\,14} = \frac{137}{278},\ a_{16,\,15} = \frac{149}{331},\ b_{21} = -\frac{47}{653},$$

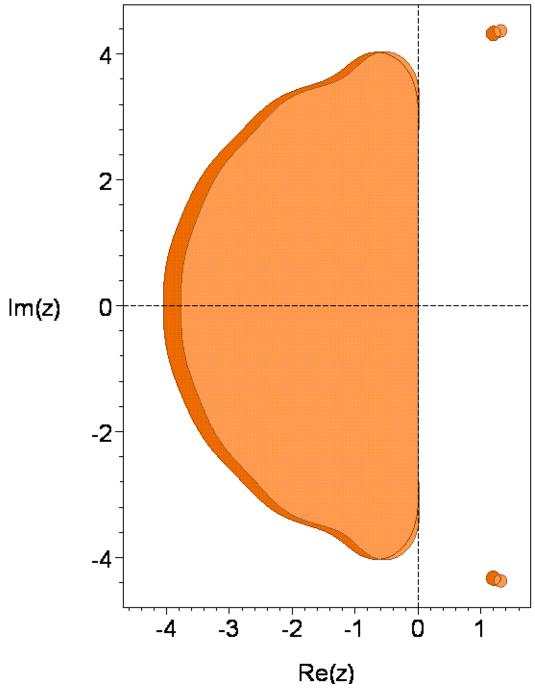
$$a_{20,\,9} = \frac{298}{1087},\ a_{20,\,12} = \frac{713}{991},\ a_{20,\,13} = -\frac{733}{557},\ a_{21,\,9} = \frac{916}{831},\ a_{21,\,13} = \frac{613}{202}.$$

The principal error norm of the order 10 scheme is $0.48197311 \times 10^{(-7)}$ and the real stability interval is $[-3.77256, 0]$.

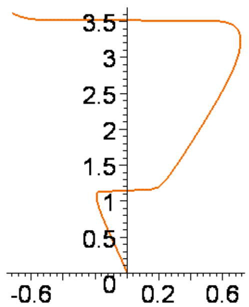
The weight that determines the order 9 embedded scheme is $b^*_{22} = -\frac{1}{19}$.

The principal error norm of the order 9 scheme is $0.25356507 \times 10^{(-6)}$ and the real stability interval is $[-4.05241, 0]$.
The maximum magnitude of the linking coefficients is 46.52264449 and the 2-norm of the linking coefficients is 97.78386946.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[1.159652\,i, 3.51651\,i]$ as well as at the origin.

4. **Scheme with a large stability region**.

$$c_7 = \frac{81}{649},\ c_{13} = \frac{241}{325},\ c_{14} = \frac{484}{1251},\ c_{15} = \frac{567}{5282},\ c_{16} = \frac{925}{1059},\ c_{17} = \frac{2764}{2875},\ c_{18} = \frac{178}{539},\ c_{19} = \frac{67}{107},\ c_{20} = \frac{716}{739},$$

$$a_{15,\,14} = -\frac{22}{145},\ a_{16,\,14} = \frac{352}{903},\ a_{16,\,15} = \frac{143}{309},\ b_{21} = -\frac{917}{6437},$$

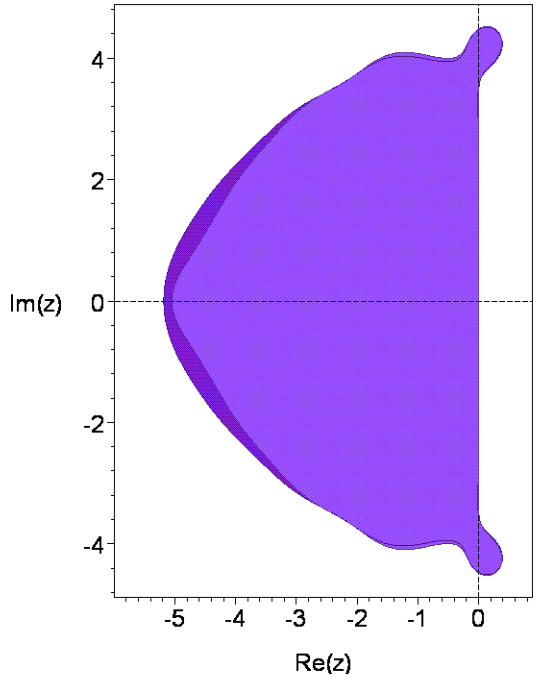
$$a_{20,\,9} = \frac{48}{197},\ a_{20,\,12} = \frac{31}{41},\ a_{20,\,13} = -\frac{163}{535},\ a_{21,\,9} = \frac{63}{46},\ a_{21,\,13} = \frac{869}{904}.$$

The principal error norm of the order 10 scheme is $0.60015882 \times 10^{(-7)}$ and the real stability interval is $[-5.05104, 0]$.

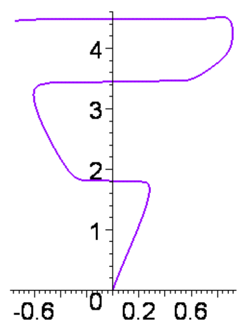
The weight that determines the order 9 embedded scheme is $b^*_{22} = -\frac{10}{87}$.

The principal error norm of the order 9 scheme is $0.31412704 \times 10^{(-6)}$ and the real stability interval is $[-5.18345, 0]$.
The maximum magnitude of the linking coefficients is 16.19434756 and the 2-norm of the linking coefficients is 43.78037143.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The intersection of the stability region with the nonnegative imaginary axis is $[0, 1.81366\,i] \cup [3.436651\,i, 4.47984\,i]$.

5. **Scheme with a moderately large stability region and a large imaginary axis inclusion**.

$$c_7 = \frac{72}{589},\ c_{13} = \frac{271}{366},\ c_{14} = \frac{324}{835},\ c_{15} = \frac{235}{2193},\ c_{16} = \frac{744}{853},\ c_{17} = \frac{417}{433},\ c_{18} = \frac{170}{517},\ c_{19} = \frac{900}{1433},\ c_{20} = \frac{213}{220},$$

$$a_{15,\,14} = -\frac{86}{561},\ a_{16,\,14} = \frac{256}{635},\ a_{16,\,15} = \frac{493}{1063},\ b_{21} = -\frac{32}{229},$$

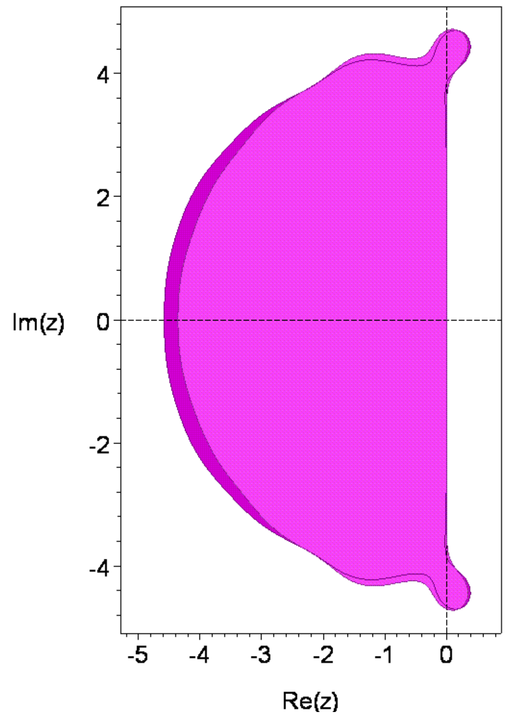
$$a_{20,\,9} = \frac{140}{603},\ a_{20,\,12} = \frac{302}{357},\ a_{20,\,13} = -\frac{150}{437},\ a_{21,\,9} = \frac{335}{228},\ a_{21,\,13} = \frac{92}{81}.$$

The principal error norm of the order 10 scheme is $0.58785712 \times 10^{(-7)}$ and the real stability interval is $[-4.35953, 0]$.

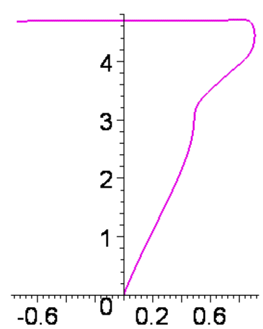
The weight that determines the order 9 embedded scheme is $b^*_{22} = -\frac{6}{53}$.

The principal error norm of the order 9 scheme is $0.30745633 \times 10^{(-6)}$ and the real stability interval is $[-4.58303, 0]$.
The maximum magnitude of the linking coefficients is 16.15043351 and the 2-norm of the linking coefficients is 45.99544634.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 4.6952\, i]$.

It is possible to construct order 10 schemes for which 617 of the 1842 principal error terms are zero. The method of construction involves performing steps 1 to 6 described previously and then proceeds with the following steps.

Step 7:
A system of equations arising from the 3 order 11 order conditions

$$\sum_{i=3}^{21} b_i\, c_i^{3} \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{6} \right) = \frac{1}{77}, \qquad \sum_{i=3}^{21} b_i\, c_i^{2} \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{7} \right) = \frac{1}{88},$$

$$\sum_{i=3}^{21} b_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{8} \right) = \frac{1}{99}.$$

can be solved to give linear expressions for the linking coefficients $a_{20,9}$, $a_{21,9}$ and $a_{21,17}$.

By means of substitution we obtain linear expressions for 49 linking coefficients in terms of the 6 linking coefficients

$$a_{20,i},\ i = 12, 13, 16, 17, \quad a_{21,i},\ i = 13, 16.$$

Step 8:
After specifying a value for $a_{20,\,13}$ the values for the remaining 5 linking coefficient parameters $a_{20,\,12}$, $a_{20,\,16}$, $a_{20,\,17}$, $a_{21,\,13}$, $a_{21,\,16}$ can be obtained by means of a system of 5 equations given by the 4 order 10 simple order conditions used in the previous step 7 together with the following additional order 11 order condition.

$$\sum_{i=10}^{21} b_i\, c_i \left(\sum_{j=9}^{i-1} a_{i,j} \left(\sum_{k=8}^{j-1} a_{j,k} \left(\sum_{l=7}^{k-1} a_{k,l} \left(\sum_{m=6}^{l-1} a_{l,m} \left(\sum_{n=5}^{m-1} a_{m,n} \left(\sum_{p=4}^{n-1} a_{n,p} \left(\sum_{q=3}^{p-1} a_{p,q} \left(\sum_{r=2}^{q-1} a_{q,r}\, c_r\right)\right)\right)\right)\right)\right)\right)\right) = \frac{1}{3991680}.$$

This system of nonlinear equations can be solved by using the multidimensional version of Newton's method if initial values are provided. As before it is usually sufficient to take the initial value for each of the variables to be zero.

The order 10 scheme is now determined by 15 parameters along with the weight b^*_{22} which determines the embedded order 9 scheme.

6. **First scheme with 617 zero error terms**.

$$c_7 = \frac{17}{50},\; c_{13} = \frac{220}{287},\; c_{14} = \frac{158}{417},\; c_{15} = \frac{30}{281},\; c_{16} = \frac{269}{297},\; c_{17} = \frac{585}{586},\; c_{18} = \frac{83}{224},\; c_{19} = \frac{954}{1547},\; c_{20} = \frac{613}{615},$$

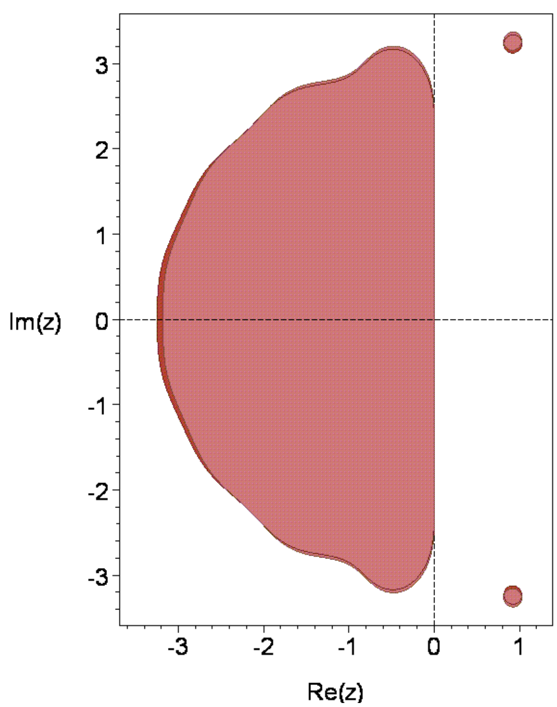
$$a_{15,\,14} = -\frac{88}{677},\; a_{16,\,14} = \frac{61}{113},\; a_{16,\,15} = \frac{188}{301},\; a_{20,\,13} = \frac{396}{53},\; b_{21} = \frac{102}{29}.$$

The principal error norm of the order 10 scheme is $0.74432991 \times 10^{(-7)}$ and the real stability interval is $[-3.18139, 0]$.

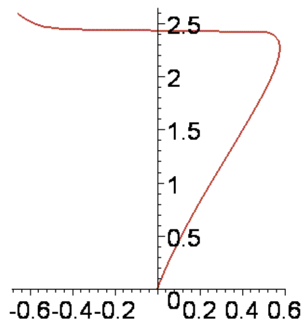
The weight that determines the order 9 embedded scheme is $b^*_{22} = \frac{17}{5}$.

The principal error norm of the order 9 scheme is $0.34899697 \times 10^{(-6)}$ and the real stability interval is $[-3.24819, 0]$.
The maximum magnitude of the linking coefficients is 9.700701189 and the 2-norm of the linking coefficients is 25.76408048.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.44538\,i]$.

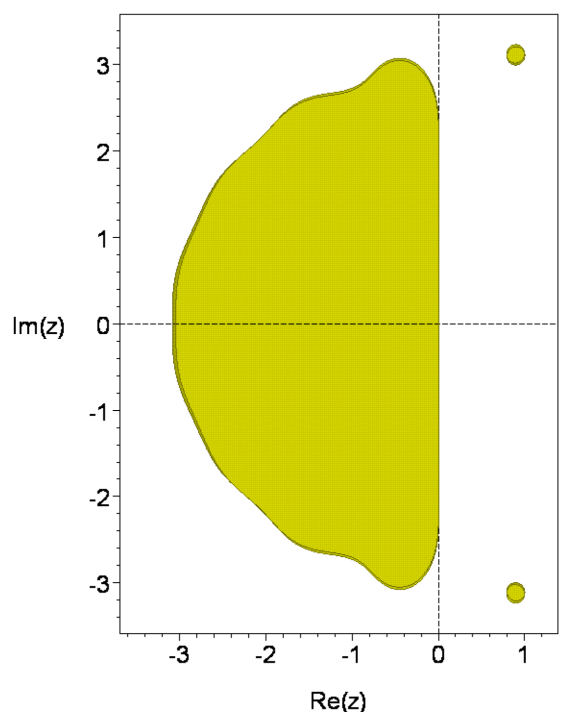
7. **Second scheme with 617 zero error terms**.

$$c_7 = \frac{179}{483},\ c_{13} = \frac{192}{247},\ c_{14} = \frac{201}{533},\ c_{15} = \frac{1163}{10855},\ c_{16} = \frac{652}{673},\ c_{17} = \frac{463}{464},\ c_{18} = \frac{833}{2256},\ c_{19} = \frac{443}{706},\ c_{20} = \frac{465}{467},$$

$$a_{15,\,14} = -\frac{22}{197},\ a_{16,\,14} = \frac{13}{344},\ a_{16,\,15} = \frac{488}{629},\ a_{20,\,13} = \frac{166}{51},\ b_{21} = \frac{98}{29}.$$

The principal error norm of the order 10 scheme is $0.65462209 \times 10^{(-7)}$ and the real stability interval is $[-3.04493, 0]$.
The weight that determines the order 9 embedded scheme is $b^*_{22} = 3$.
The principal error norm of the order 9 scheme is $0.32821685 \times 10^{(-6)}$ and the real stability interval is $[-3.08514, 0]$.
The maximum magnitude of the linking coefficients is 7.413078774 and the 2-norm of the linking coefficients is 16.05022843.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.37439\,i]$.

It is possible to construct order 10 schemes for which 681 of the 1842 principal error terms are zero. The method of construction involves performing steps 1 to 7 given for schemes that have 617 zero error terms.
Step 8 is altered to include the order 11 order condition

$$\sum_{i=4}^{21} b_i\, c_i^3 \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^5 \right) \right) = \frac{1}{462}$$

instead of the one given in the previous step 8. The order 10 scheme is determined by 15 parameters along with the weight b^*_{22} which determines the embedded order 9 scheme.

8. **First scheme with 681 zero error terms**.

$$c_7 = \frac{493}{1716},\ c_{13} = \frac{138}{179},\ c_{14} = \frac{526}{1387},\ c_{15} = \frac{95}{861},\ c_{16} = \frac{70}{79},\ c_{17} = \frac{247}{253},\ c_{18} = \frac{424}{891},\ c_{19} = \frac{172}{287},\ c_{20} = \frac{6957}{7031},$$

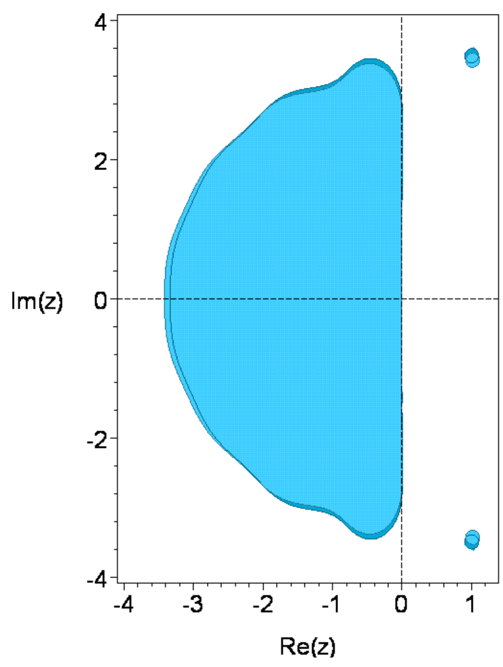
$$a_{15,14} = -\frac{409}{3002},\ a_{16,14} = \frac{157}{320},\ a_{16,15} = \frac{154}{305},\ a_{20,13} = \frac{497}{158},\ b_{21} = \frac{351}{280}.$$

The principal error norm of the order 10 scheme is $0.84515433 \times 10^{(-7)}$ and the real stability interval is $[-3.41460, 0]$.

The weight that determines the order 9 embedded scheme is $b^*_{22} = \frac{38}{31}$.

The principal error norm of the order 9 scheme is $0.40866930 \times 10^{(-6)}$ and the real stability interval is $[-3.33341, 0]$.
The maximum magnitude of the linking coefficients is 8.190755066 and the 2-norm of the linking coefficients is 20.12037956.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.72227\, i]$.

9. **Second scheme with 681 zero error terms**.

$$c_7 = \frac{313}{849},\ c_{13} = \frac{1137}{1438},\ c_{14} = \frac{557}{1458},\ c_{15} = \frac{230}{2049},\ c_{16} = \frac{396}{439},\ c_{17} = \frac{681}{692},\ c_{18} = \frac{232}{481},\ c_{19} = \frac{46}{75},\ c_{20} = \frac{241}{242},$$

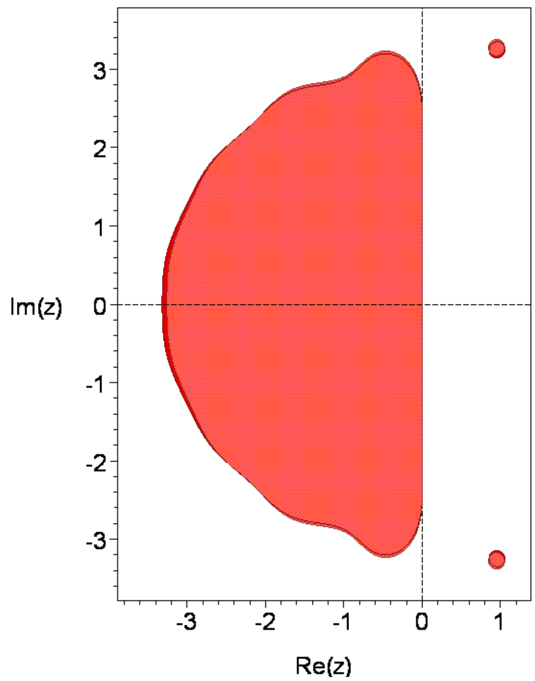
$$a_{15,14} = -\frac{166}{1383},\ a_{16,14} = \frac{171}{341},\ a_{16,15} = \frac{276}{545},\ a_{20,13} = \frac{18}{521},\ b_{21} = \frac{592}{155}.$$

The principal error norm of the order 10 scheme is $0.55428759 \times 10^{(-7)}$ and the real stability interval is $[-3.25659, 0]$.

The weight that determines the order 9 embedded scheme is $b^*_{22} = \frac{77}{20}$.

The principal error norm of the order 9 scheme is $0.26751874 \times 10^{(-6)}$ and the real stability interval is $[-3.31718, 0]$.
The maximum magnitude of the linking coefficients is 5.860259802 and the 2-norm of the linking coefficients is 14.32380140.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.53201\, i]$.

Version: 6 Feb 2015, Peter Stone